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Canonical Transformations of Dirac's Equation to Even Forms. Expansion in Terms of the External Fields.

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1. – Introduction.

The motion of a fermion in various types of external fields is described by a four-component wave function ψ, satisfying the Dirac equation

$$i\frac{\partial\psi}{\partial t} = H\psi = (H_0 + gH_1)\psi\,.$$

H_0 is the free particle Hamiltonian,

$$H_0 = \beta m + \boldsymbol{\alpha}\boldsymbol{p},$$

where β and α are the well known Dirac matrices, and units are chosen so that $\hbar = c = 1$. The representation of β and α is the usual one, with β diagonal: $\beta_{11} = \beta_{22} = 1$, $\beta_{33} = \beta_{44} = -1$. The interaction with the external field is represented by the term gH_1, where g is the coupling constant, for example the electric charge of the fermion in the case of external electromagnetic fields.

Following FOLDY and WOUTHUYSEN [1], we distinguish between even and odd operators. An odd operator in the Dirac theory is a Dirac matrix which has only matrix elements connecting upper and lower components of the wave function, while an even operator is one having no such matrix elements. It can easily be shown that a necessary and sufficient condition for a matrix to be even (odd) is that it commutes (anticommutes) with β. This allows one to write any matrix, say M, as a sum of an even and an odd matrix in a simple way:

$$M = M^e + M^o ,$$

where

$$M^e = \tfrac{1}{2}(M + \beta M \beta) = \text{even part} ,$$

$$M^o = \tfrac{1}{2}(M - \beta M \beta) = \text{odd part} .$$

If the Hamiltonian were an even operator, the Dirac equation would separate into 2 two-component wave equations, one equation for the upper components, and one for the lower components. In their paper FOLDY and WOUTHUYSEN demonstrate how the Hamiltonian, by means of successive canonical transformations, can be converted to an operator, which is even to any desired order in the expansion parameter $1/m$. The Hamiltonian is put in the form

$$H = \beta m + \mathscr{E} + \mathscr{O} ,$$

where $\mathscr{E}$ is even and $\mathscr{O}$ is odd. Then a canonical transformation is performed,

$$\psi' = \exp[iS]\psi ,$$

$$H' = \exp[iS]\left(H - i\frac{\partial}{\partial t}\right)\exp[-iS] + i\frac{\partial}{\partial t} =$$

$$= H + i\left[S, H - i\frac{\partial}{\partial t}\right] - \frac{1}{2}\left[S, \left[S, H - i\frac{\partial}{\partial t}\right]\right] + \ldots .$$

With $S = -i\beta\mathscr{O}/2m$, the lowest order odd term in $1/m$ will be of first order, *i.e.* of one order higher than in the original Hamiltonian. By a sequence of further transformations, the generator of the transformation at each step being

(1) L. L. FOLDY and S. WOUTHUYSEN: *Phys. Rev.*, **78**, 29 (1950).

chosen as $S = -(i\beta/2m) \times$ (odd terms in the Hamiltonian of lowest order in $1/m$), the Hamiltonian can be made even to any desired order in $1/m$. The resulting transformation is the product

$$(1) \qquad U_{\mathrm{FW}} = \ldots \exp[iS_2] \exp[iS_1],$$

$$S_1 = -(i/2m)\beta \mathcal{O}, \qquad S_2 = -(i/4m^2)\left[\mathcal{O}, \mathcal{E} - i\frac{\partial}{\partial t}\right],$$

and the transformed Hamiltonian to third order in $1/m$ is

$$H^{\mathrm{tr}} = \beta m + \mathcal{E} + (1/2m)\beta \mathcal{O}^2 - (1/8m^2)\left[\mathcal{O}, \left[\mathcal{O}, \mathcal{E} - i\frac{\partial}{\partial t}\right]\right] -$$

$$- (1/8m^3)\beta\left[\mathcal{O}, \mathcal{E} - i\frac{\partial}{\partial t}\right]^2 - (1/8m^3)\beta \mathcal{O}^4 + \ldots .$$

When the particle is free, the transformation can be written in closed form,

$$U_0 = \exp[iS],$$

where S is odd and Hermitian,

$$S = -(i/2p)\beta \boldsymbol{\alpha p} \operatorname{tg}^{-1}(p/m), \qquad p = \sqrt{\boldsymbol{p}^2}$$

The transformed Hamiltonian becomes

$$H_0^{\mathrm{tr}} = \beta E_p = \beta\sqrt{m^2 + p^2}.$$

A canonical transformation in closed form, making the Hamiltonian even, by time-independent external fields, is given by ERIKSEN [2]. He considers the equation

$$[\beta, H^{\mathrm{tr}}] = [\beta, UHU^*] = 0,$$

which is a necessary and sufficient condition for H^{tr} to be even. The condition is equivalent to

$$[U^*\beta U, H] = 0.$$

A possible choice for $U^*\beta U$ is

$$(2) \qquad U^*\beta U = H/\sqrt{H^2} \equiv \lambda,$$

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[2] E. ERIKSEN: *Phys. Rev.*, **111**, 1011 (1958).

i.e.

$$\beta = U\lambda U^* = H^{\mathrm{tr}}/\sqrt{(H^{\mathrm{tr}})^2}\,,$$

or

$$H^{\mathrm{tr}} = \beta\sqrt{(H^{\mathrm{tr}})^2}\,.$$

This means that the transformation makes the Hamiltonian even and energy-separating. That is: the wave equation for the upper components of ψ^{tr} has positive energy eigenvalues only, and the wave equation for the lower components has negative energy eigenvalues only. Obviously, the condition (2) is also a necessary one, for the Hamiltonian to be even and energy-separating.

As a solution of eq. (2) ERIKSEN takes

$$U = \sqrt{\beta\lambda}\,.$$

This is a square root of a unitary operator:

$$U = \sqrt{u}\,, \qquad uu^* = u^*u = 1\,, \qquad u = \beta\lambda\,.$$

For the definition of this symbol we refer to the Appendix.

U has the property (see Appendix),

$$\beta U = U^*\beta\,,$$

which implies that U can be written as

$$U = \exp\left[iS\right],$$

where S is *odd* and Hermitian. This shows that U, in the general case, is different from U_{FW} given by (1). According to a theorem by HAUSSDORFF [3] for the product of « symbolic » exponential functions,

$$\exp\left[A\right]\exp\left[B\right] = \exp\left[A + B + \tfrac{1}{2}[A, B] + \text{higher order commutators}\right],$$

we have to third order in $1/m$

$$U_{\mathrm{FW}} = \exp\left[iS_1 + iS_2 + iS_3 + \tfrac{1}{2}[S_1, S_2] + \ldots\right].$$

$[S_1, S_2]$ is an *even* operator, and generally non-vanishing. Hence, the transformations are in general different. They also produce different Hamiltonians.

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[3] F. HAUSSDORFF: *Ber. Sächs. Ges.*, **58**, 19 (1906).

To lowest order in $1/m$ the difference is $(1/16m^3)\beta[\mathscr{E}, [\mathscr{E}, \mathscr{O}^2]]$, which should be added to $H_{\mathrm{FW}}^{\mathrm{tr}}$ in order to give Eriksen's result. When the particle is free, the transformations are equal.

When the interaction term is odd, *i.e.*

$$H = \beta m + \mathscr{O} ,$$

the transformation with $\sqrt{\beta\lambda}$ gives a simple result:

$$H^{\mathrm{tr}} = \beta\sqrt{m^2 + \mathscr{O}^2} .$$

Applying this to an electron in a magnetostatic field, $\boldsymbol{H} = \mathrm{curl}\ \boldsymbol{A}$, we get

$$H^{\mathrm{tr}}(\Phi = 0) = \beta[m^2 + (\boldsymbol{p} - e\boldsymbol{A})^2 - e\boldsymbol{\sigma H}]^{\frac{1}{2}} , \tag{3}$$

which result has also been obtained by CASE [4], by a different approach.

When, however, an external electric field is present, there is no simple form for the transformed Hamiltonian. Neither is there any simple expansion in powers of the electric charge. In general, if one wants to have H^{tr} expanded in powers of some coupling parameter, one will find extremely complicated results already in the first order term. This is due to the fact that the zero order term of $\beta\lambda$ (the square root of which is U) is no c-number quantity, but equal to the operator $(m+\beta\boldsymbol{\alpha p})E_p^{-1}$.

The main purpose of the present paper is to introduce various types of canonical transformations, which, apart from making the Hamiltonian even and energy separating, have the property that they can be expanded in powers of the coupling constant as simply as possible.

Since the condition $U^*\beta U = \lambda$ is a necessary and sufficient one in order to have the Hamiltonian even and energy-separating, we shall first consider the operator λ, and expand it in powers of the coupling constant g. An expansion of λ in powers of $1/m$ will also be made.

Then we shall present three unitary operators in closed form, satisfying the equation $U^*\beta U = \lambda$, and which are appropriate for an expansion in powers of g. Two of them are in the free particle case different from the Foldy-Wouthuysen transformation function, and in some respect of a simpler nature.

Transformations in powers of g will also be found by an iteration procedure analogous to the Foldy-Wouthuysen method. In this connection we derive advantage from a pleasant feature of the iteration method, namely that the transformed Hamiltonian to order $2n+1$ is being expressed in terms of the n first generators of the transformation.

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[4] K. M. CASE: *Phys. Rev.*, **95**, 1323 (1954).

Then we shall proceed to the general case with explicitly time dependent external fields, and it will be shown that the generalization mainly consists in replacing λ by an operator Λ, which is a constant of motion in the Dirac picture.

The transformed Hamiltonians to first order in e will be written out for time dependent external electromagnetic fields. Certain terms of first order, which can be expressed as commutators with βE_p, can be removed to first order by an additional even unitary transformation. The transition to the classical case is performed. As an example, the Born approximation in one of the new representations (pictures) is calculated.

In the new representations the Dirac particle shows, in a way, a behaviour characteristic of a particle with extension. The charge distribution of this extended particle, in the various representations, is expressed in terms of well known functions.

A) Time Independent External Fields.

2. – Expansions of the operator λ.

2·1. *Expansions in powers of g.* – We shall make the series expansion

$$\lambda \equiv H/\sqrt{H^2} = \lambda_0 + g\lambda_1 + g^2\lambda_2 + \dots ,$$

where

$$H = H_0 + gH_1 .$$

We obviously have

$$\lambda^2 = 1 , \tag{4}$$

$$[\lambda, H_0 + gH_1] = 0 , \tag{5}$$

$$\lambda_0 = H_0 E_p^{-1} . \tag{6}$$

As will be seen below these properties are sufficient to determine the expansion coefficients λ_n.

In the following we will make use of the fact that any operator F can be written as the sum of two parts, one part (F^c) commuting with λ_0, and one part (F^a) anticommuting with λ_0:

$$\begin{cases} F = F^c + F^a , \\ F^c = \frac{1}{2}(F + \lambda_0 F \lambda_0) , \\ F^a = \frac{1}{2}(F - \lambda_0 F \lambda_0) . \end{cases} \tag{7}$$

According to (4)

$$(8)\qquad \left|\begin{array}{l} (\lambda_0 + g\lambda_1 + g^2\lambda_2 + \ldots)^2 = 1\,, \\ i.e. \\ \lambda_0\lambda_1 + \lambda_1\lambda_0 = 0\,, \\ \lambda_0\lambda_2 + \lambda_1^2 + \lambda_2\lambda_0 = 0\,, \quad \text{etc.} \end{array}\right.$$

By means of these relations λ_n^c can be expressed in terms of the λ_i's of lower order

$$\lambda_1^c = 0\,,$$

$$\lambda_2^c = -\tfrac{1}{2}\lambda_0\lambda_1^2\,, \quad \text{etc.}$$

or generally

$$(9)\qquad \lambda_n^c = -\tfrac{1}{2}\lambda_0 \sum_{k=1}^{n-1} \lambda_k\, \lambda_{n-k}.$$

The anticommuting parts are determined by (5), which is equivalent to

$$[\lambda_n, H_0] + [\lambda_{n-1}, H_1] = 0\,, \qquad n = 1, 2, \ldots.$$

The anticommuting part of this equation is

$$[\lambda_n^a, H_0] + [\lambda_{n-1}, H_1]^a = 0\,.$$

Putting $H_0 = \lambda_0 E_p$ and multiplying by λ_0, we get

$$(10)\qquad [\lambda_n^a, E_p]_+ = b_n^a\,, \qquad b_n = \lambda_0[\lambda_{n-1}, H_1]\,.$$

This equation has one solution only, since the homogeneous equation is only satisfied by $\lambda_n^a = 0$ (E_p being positive definite).

It is possible to write λ_n^a in the following way:

$$(11)\qquad \lambda_n^a = \int_{-\infty}^{0} \exp[E_p\tau]\, b_n^a \exp[E_p\tau]\, d\tau\,.$$

This way of writing operators has also been used by MORPURGO [5] (eq. (51)

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[5] G. MORPURGO: *Nuovo Cimento*, **15**, 624 (1960).

in his paper). The proof is simple:

$$[\lambda_n^a, E_p]_+ = \int_{-\infty}^{0} \exp[E_p\tau](E_p b_n^a + b_n^a E_p)\exp[E_p\tau]\,d\tau = \int_{-\infty}^{0} d(\exp[E_p\tau] b_n^a \exp[E_p\tau]) = b_n^a .$$

Hence, λ_n can be expressed as follows

$$\lambda_n = -\frac{1}{2}\lambda_0 \sum_{k=1}^{n-1} \lambda_k \lambda_{n-k} + \frac{1}{2}\int_{-\infty}^{0} \exp[E_p\tau][\lambda_0, [\lambda_{n-1}, H_1]]\exp[E_p\tau]\,d\tau , \tag{12}$$

where the sum is zero when $n = 1$:

$$\lambda_1 = \lambda_0 \int_{-\infty}^{0} \exp[E_p\tau][\lambda_0, H_1]\exp[E_p\tau]\,d\tau . \tag{13}$$

For the evaluation of these integrals it may be convenient to make the following transformation:

$$\boldsymbol{r}' = \exp[E_p\tau]\boldsymbol{r}\exp[-E_p\tau] = \boldsymbol{r} + \tau[E_p, \boldsymbol{r}] + \ldots = \boldsymbol{r} + i\tau[E_p, \partial/\partial\boldsymbol{p}] = \boldsymbol{r} - i\tau E_p^{-1}\boldsymbol{p} ,$$

i.e.

$$\exp[E_p\tau]F(\boldsymbol{r}) = \exp[E_p\tau]F(\boldsymbol{r})\exp[-E_p\tau]\exp[E_p\tau] = F(\boldsymbol{r}')\exp[E_p\tau] .$$

Hence

$$\lambda_1 = \lambda_0 \int_{-\infty}^{0} [\lambda_0, H_1(\boldsymbol{r}')]\exp[2E_p\tau]\,d\tau .$$

Example a). Electron in a constant electric field $\boldsymbol{E}$, *i.e.* $H_1 = -\boldsymbol{E}\boldsymbol{r}$:

$$\begin{cases} [\lambda_0, H_1(\boldsymbol{r}')] = [\lambda_0, -\boldsymbol{E}\boldsymbol{r} + i\tau E_p^{-1}\boldsymbol{E}\boldsymbol{p}] = -\boldsymbol{E}[\lambda_0, \boldsymbol{r}] = i\boldsymbol{E}\dfrac{\partial\lambda_0}{\partial\boldsymbol{p}} , \\ \lambda_1 = i\lambda_0\boldsymbol{E}\dfrac{\partial\lambda_0}{\partial\boldsymbol{p}}\displaystyle\int_{-\infty}^{0}\exp[2E_p\tau]\,d\tau = \frac{1}{2}i\lambda_0 E_p^{-1}\boldsymbol{E}\frac{\partial\lambda_0}{\partial\boldsymbol{p}} = \\ \qquad = \dfrac{1}{2}E_p^{-3}(im\beta\boldsymbol{\alpha}\boldsymbol{E} - \boldsymbol{\sigma}(\boldsymbol{p}\times\boldsymbol{E})) . \end{cases} \tag{14}$$

Example b). Fourier component: $H_1(\boldsymbol{k}) = \exp[i\boldsymbol{kr}]$.

$$\exp[E_p\tau]\exp[i\boldsymbol{kr}] = \exp[i\boldsymbol{kr}]\exp[-i\boldsymbol{kr}]\exp[E_p\tau]\exp[i\boldsymbol{kr}] =$$
$$= \exp[i\boldsymbol{kr}]\exp[E'_p\tau], \qquad E'_p = [m^2 + (\boldsymbol{p}+\boldsymbol{k})^2]^{\frac{1}{2}}.$$

Here we have used

$$\exp[-i\boldsymbol{kr}]\boldsymbol{p}\exp[i\boldsymbol{kr}] = \boldsymbol{p} + \boldsymbol{k}\,.$$

Hence

$$(15)\quad \begin{cases} I(\boldsymbol{k}) \equiv \int_{-\infty}^{0} \exp[E_p\tau]\exp[i\boldsymbol{kr}]\exp[E_p\tau]\,\mathrm{d}\tau = \\ \qquad = \exp[i\boldsymbol{kr}]\int_{-\infty}^{0}\exp[(E'_p + E_p)\tau]\,\mathrm{d}\tau = \exp[i\boldsymbol{kr}](E'_p + E_p)^{-1}\,, \\ \lambda_1(\boldsymbol{k}) = I(\boldsymbol{k}) - \lambda_0 I(\boldsymbol{k})\lambda_0 = \\ \qquad = \exp[i\boldsymbol{kr}]\{1 - \exp[-i\boldsymbol{kr}]\lambda_0\exp[i\boldsymbol{kr}]\lambda_0\}(E'_p + E_p)^{-1} = \\ \qquad\qquad = \exp[i\boldsymbol{kr}](1 - \lambda'_0\lambda_0)(E'_p + E_p)^{-1}\,, \end{cases}$$

where $\lambda'_0 = (H_0 + \boldsymbol{\alpha k})/E'_p$.

Example c). Oscillator potential: $H_1 = r^2$:

$$\exp[E_p\tau][\lambda_0, \boldsymbol{rr}]\exp[E_p\tau] = [\lambda_0, (\boldsymbol{r} - i\tau E_p^{-1}\boldsymbol{p})\exp[2E_p\tau](\boldsymbol{r} + i\tau E_p^{-1}\boldsymbol{p})]$$
$$= [\lambda_0, \boldsymbol{r}\exp[2E_p\tau]\boldsymbol{r} - i\tau[E_p^{-1}\boldsymbol{p}\exp[2E_p\tau], \boldsymbol{r}]]$$
$$= [\lambda_0, \boldsymbol{r}\exp[2E_p\tau]\boldsymbol{r}]\,,$$

$$\lambda_1 = \lambda_0\int_{-\infty}^{0}[\lambda_0, \boldsymbol{r}\exp[2E_p\tau]\boldsymbol{r}]\,\mathrm{d}\tau = \frac{1}{2}\lambda_0[\lambda_0, \boldsymbol{r}E_p^{-1}\boldsymbol{r}]\,.$$

2·2. *Expansion in powers of* $1/m$. – In order to find the coefficients $\lambda_{(n)}$ in the series

$$\lambda = \lambda_{(0)} + (1/m)\lambda_{(1)} + (1/m)^2\lambda_{(2)} + \dots\,,$$

we use the same technique as above:

$$(16)\qquad \lambda^2 = 1\,,$$

$$(17)\qquad [\lambda, H] = 0\,, \qquad H = m\beta + h = m\beta + \mathscr{E} + \mathscr{O},$$

$$(18)\qquad \lambda_{(0)} = \beta\,.$$

By means of (16) the even part of $\lambda_{(n)}$ can be expressed in terms of the coefficients of lower order

$$\lambda^{\mathrm{e}}_{(n)} = -\tfrac{1}{2}\beta \sum_{k=1}^{n-1} \lambda_{(k)} \lambda_{(n-k)} , \qquad \lambda^{\mathrm{e}}_{(1)} = 0.$$

The odd part is determined by (17), which gives the following equations

$$[\lambda_{(n)}, \beta] + [\lambda_{(n-1)}, h] = 0 , \qquad n = 1, 2, \ldots,$$

i.e.

$$\lambda^{\mathrm{o}}_{(n)} = \tfrac{1}{2}\beta[\lambda_{(n-1)}, h] .$$

Hence

$$\lambda_{(n)} = \lambda^{\mathrm{o}}_{(n)} + \lambda^{\mathrm{e}}_{(n)} = \tfrac{1}{2}\beta\Big([\lambda_{(n-1)}, h] - \sum_{k=1}^{n-1} \lambda_{(k)} \lambda_{(n-k)}\Big) . \tag{19}$$

The lowest order terms are

$$\lambda_{(1)} = \mathcal{O} , \quad \lambda_{(2)} = \tfrac{1}{2}\beta[\mathcal{O}, \mathcal{E}] - \tfrac{1}{2}\beta\mathcal{O}^2 ,$$

$$\lambda_{(3)} = \tfrac{1}{4}[\mathcal{E}, [\mathcal{E}, \mathcal{O}]] + \tfrac{1}{4}[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] - \tfrac{1}{2}\mathcal{O}^3 .$$

We are now in a position to make a formal proof that the Foldy-Wouthuysen transformation gives an energy separating representation in the case of non explicitly time dependent external fields. Consider the operator $U^*_{\mathrm{FW}}\beta U_{\mathrm{FW}} (\equiv Z)$. We obviously have

$$Z^2 = 1 \quad \text{and} \quad Z = \beta + (1/m) Z_{(1)} + \ldots .$$

Further, we know that $H^{\mathrm{tr}}_{\mathrm{FW}}$ is even, *i.e.*

$$[\beta, U_{\mathrm{FW}} H U^*_{\mathrm{FW}}] = 0 ,$$

or

$$[Z, H] = 0 .$$

Thus Z satisfies the equations which define λ, *i.e.*

$$Z \equiv U^*_{\mathrm{FW}}\beta U_{\mathrm{FW}} = \lambda ,$$

but this is just the condition which U must satisfy in order to give an energy separating representation.

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3. – Transformations in closed form.

As mentioned in the introduction, a necessary and sufficient condition for the transformed Hamiltonian, $H^{\mathrm{tr}} = UHU^*$, to be even and energy separating is

$$U\lambda U^* = \beta\,, \qquad \text{where } UU^* = U^*U = 1. \tag{20}$$

A solution of this equation is the operator $U = \sqrt{\beta\lambda}$, which, however, gives extremely complicated terms when expanded in powers of g.

Our problem is to find a transformation which brings λ over into β. Our device is that this can be effected in two (or more) steps, by introducing an intermediate operator f, $f = f^* = f^{-1}$, so that λ transforms into β according to the scheme

$$\lambda \to f \to \beta\,.$$

The first transformation, $\lambda \to f$, can be carried out by means of the transformation function $\sqrt{f\lambda}$,

$$\sqrt{f\lambda}\;\lambda\sqrt{\lambda f} = f\,,$$

and the second transformation, $f \to \beta$, correspondingly by

$$\sqrt{\beta f}\;f\sqrt{f\beta} = \beta\,.$$

The resulting transformation, being the product of two unitary transformations, is of course also unitary:

$$U = \sqrt{\beta f}\,\sqrt{f\lambda}\,, \tag{21}$$

$$U\lambda U^* = \sqrt{\beta f}\,\sqrt{f\lambda}\,\lambda\sqrt{\lambda f}\,\sqrt{f\beta} = \sqrt{\beta f}\,f\,\sqrt{f\beta} = \beta\,.$$

Obviously the method can be generalized to an arbitrary number of transformations,

$$U = \sqrt{\beta f_n}\,\sqrt{f_n f_{n-1}}\,\ldots\,\sqrt{f_2 f_1}\,\sqrt{f_1\lambda}\,.$$

Now consider the definition (Appendix) of $\sqrt{f\lambda}$,

$$\sqrt{f\lambda} = (1 + f\lambda)(2 + f\lambda + \lambda f)^{-\frac{1}{2}}\,. \tag{22}$$

A simple expansion in powers of g (when expressed in terms of λ_i) can be obtained by requiring the zero order term in the square root to be a c-number

quantity. Letting f be of zero order, we require

$$f\lambda_0 + \lambda_0 f = c\text{-number}\,, \qquad f = f^* = f^{-1}\,. \tag{23}$$

3·1. *Transformation I.* – As is well known there exists a Dirac matrix,

$$\eta = i\beta\gamma_5 = -\beta\alpha_1\alpha_2\alpha_3\,, \tag{24}$$

with the properties

$$\eta = \eta^* = \eta^{-1}\,, \quad [\eta, \beta]_+ = 0\,, \quad [\eta, \boldsymbol{\alpha}]_- = 0\,.$$

Hence this matrix is a possible choice for f in eq. (23), where the c-number quantity then becomes zero,

$$[\eta, \lambda_0]_+ = [\eta, \beta m + \boldsymbol{\alpha p}]_+ E_p^{-1} = 0\,.$$

The corresponding transformation in two steps is

$$U_{\mathrm{I}} = \sqrt{\beta\eta}\,\sqrt{\eta\lambda}\,, \tag{25}$$

which transforms λ into β via the Dirac matrix η. Due to the relation $\beta\eta + \eta\beta = 0$, the square root $\sqrt{\beta\eta}$ is simply

$$\sqrt{\beta\eta} = \frac{1}{\sqrt{2}}(1 + i\gamma_5)\,,$$

i.e.

$$U_{\mathrm{I}} = \frac{1}{\sqrt{2}}(1 + i\gamma_5)\,\sqrt{\eta\lambda}\,.$$

To first order in the coupling constant,

$$U_{\mathrm{I}} = \tfrac{1}{2}(1 + i\gamma_5)(1 + \eta\lambda_0)\,[1 + g(\tfrac{1}{4}[\eta, \lambda_1] + \tfrac{1}{2}\lambda_0\lambda_1) + \ldots]\,. \tag{26}$$

In the limit of vanishing external fields,

$$U_{\mathrm{I}} \to V_0 \equiv \tfrac{1}{2}(1 + i\gamma_5)(1 + \eta\lambda_0) = \tfrac{1}{2}(1 + i\gamma_5)E_p^{-1}(E_p + \eta H_0)\,. \tag{27}$$

It should be noticed that this transformation does not coincide with the Foldy-Wouthuysen result, which can be written as

$$U_0 = [2E_p(E_p + m)]^{-\frac{1}{2}}(E_p + \beta H_0)\,. \tag{28}$$

The transformed Hamiltonian is in either case equal to βE_p.

In order to make calculations and expressions simpler, we rather express U_I in the following way:

$$U_I = \sqrt{\beta\eta}\,\sqrt{\eta\lambda}\,\sqrt{\eta\beta}\,\sqrt{\beta\eta} = \sqrt{\beta\lambda'}\,\sqrt{\beta\eta}\,.$$

Here, and in the following, we let a dash on a symbol denote the symbol transformed by $\sqrt{\beta\eta}$,

$$F' \equiv \sqrt{\beta\eta}\,F\,\sqrt{\eta\beta} = \tfrac{1}{2}(1 + i\gamma_5)\,F(1 - i\gamma_5)\,, \tag{29}$$

i.e.

$$\eta' = \beta\,, \quad \beta' = -\eta\,, \quad H_0' = -\eta m + \boldsymbol{\alpha p}\,.$$

Then the transformed result can be interpreted as

$$H_I^{tr} = \sqrt{\beta\lambda'}\,H'\,\sqrt{\lambda'\beta}\,, \quad H' = \sqrt{\beta\eta}\,H\,\sqrt{\eta\beta}\,.$$

The transformation from H to H' means going over to a Dirac matrix representation whose characteristic property is to make the free particle Hamiltonian an odd operator. The next transformation, $H' \to H^{tr}$, is in fact Eriksen's transformation, $U_E = \sqrt{\beta\bar{\lambda}}$, in this representation.

In this notation, and to first order in the coupling constant, we get

$$\sqrt{\beta\lambda'} = \frac{1}{\sqrt{2}}(1 + \beta\lambda_0') + \frac{1}{\sqrt{2}}\,g\,(\tfrac{1}{2}\lambda_0'\lambda_1'^{e} - \tfrac{1}{2}\beta\lambda_1'^{e} + \beta\lambda_1')\,, \tag{30}$$

$$H_I^{tr} = \beta E_p + \tfrac{1}{2}g(\lambda_0'[\lambda_0', H_1'^{e}]_+ + \beta[\lambda_0', H_1'^{o}]_+ + \beta[\lambda_1'^{o}, H_0']_+) + \dots\,. \tag{31}$$

As usual the indices e and o denote even and odd part, respectively.

There are two special cases in which the Hamiltonian H_I^{tr} is considerably simplified:

a) H_1' is even. This is equivalent to

$$[H_1', \beta] = 0\,, \quad \textit{i.e.} \quad [H_1, \eta] = [H_1, i\beta\gamma_5] = 0\,.$$

According to (13), λ_1' is given by

$$\lambda_1' = \lambda_0' \int_{-\infty}^{0} \exp\,[E_p\tau]\,[\lambda_0', H_1']\,\exp\,[E_p\tau]\,\mathrm{d}\tau\,,$$

which is an even operator, since λ_0' is odd. Hence $\lambda_1'^{o} = 0$, and we have

$$H_I^{tr} = \beta E_p + \tfrac{1}{2}g(H_1' + \lambda_0' H_1 \lambda_0') + \dots\,. \tag{32}$$

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b) H_1' is odd. Then the Hamiltonian H' is odd,

$$[H', \beta]_+ = 0\,, \qquad i.e. \qquad [H, \eta]_+ = [H, i\beta\gamma_5]_+ = 0\,.$$

Thus, the characteristic feature in this case is the fact that there exists a Dirac matrix which anticommutes with the Hamiltonian. Since λ' is odd, we get

$$(33) \qquad \begin{cases} \sqrt{\beta\lambda'} = (1 + \beta\lambda')(2 + \beta\lambda' + \lambda'\beta)^{-\frac{1}{2}} = \dfrac{1}{\sqrt{2}}\,(1 + \beta\lambda')\,, \\ H_1^{tr} = \frac{1}{2}(1 + \beta\lambda')\,H'\,(1 - \beta\lambda') = \frac{1}{2}(1 + \beta\lambda')^{\cdot}\,H'\,, \\ \qquad - \beta\lambda' H' = \beta\sqrt{H'^2}\,, \end{cases}$$

where $H' = \frac{1}{2}(1 + i\gamma_5)\,H(1 - i\gamma_5)$.

Consider the case of a fermion with an anomalous magnetic moment μ in a magnetostatic field $\boldsymbol{H} = \operatorname{curl} \boldsymbol{A}$,

$$\mu = \mu_D + \mu_I\,.$$

μ_D is the Dirac part $= e/2m$, and μ_I is the intrinsic part of the magnetic moment. Then one can use a Hamiltonian amplified with a Pauli interaction term,

$$H = \beta m + \boldsymbol{\alpha}(\boldsymbol{p} - e\boldsymbol{A}) - \mu_I \beta \boldsymbol{\sigma H}\,,$$

which is an example of a Hamiltonian which anticommutes with $i\beta\gamma_5$,

$$H' = -\eta m + \boldsymbol{\alpha}(\boldsymbol{p} - e\boldsymbol{A}) + \mu_I \eta \boldsymbol{\sigma H}\,,$$

$$H_1^{tr} = \beta[m^2 + (\boldsymbol{p} - e\boldsymbol{A})^2 - 2m\mu\boldsymbol{\sigma H} + \mu_I\beta\boldsymbol{\sigma}(\boldsymbol{H} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{H}) + \mu_I^2 \boldsymbol{H}^2]^{\frac{1}{2}}\,.$$

When $\mu_I = 0$, this is the same result as given by eq. (3), which was obtained by a different transformation.

Because of the term $\frac{1}{2}g\beta[\lambda_1'^{\prime\prime}, H_0']_+$ the transformed Hamiltonian to first order in the general case (31) is by no means any simple operator. One should, however, notice that the term can be written as

$$\tfrac{1}{2}g\beta[\lambda_1'^{\prime\prime}, H_0']_+ = [\tfrac{1}{2}g\lambda_1'^{\prime\prime}\lambda_0', \beta E_p]\,,$$

which is possible due to the relations $[\lambda_0', \beta]_+ = 0$ and $[\lambda_0', \lambda_1'^{\prime\prime}]_+ = 0$. Thus, to first order the term can be removed from the Hamiltonian by means of an additional even unitary transformation

$$G = \exp[-\tfrac{1}{2}g\lambda_1'^{\prime\prime}\lambda_0']\,,$$

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giving the Hamiltonian

$$GH_{\mathrm{I}}^{\mathrm{tr}}G^{*} = H_{\mathrm{I}}^{\mathrm{tr}} - \tfrac{1}{2}g[\lambda_1'^{\mathrm{o}}\lambda_0', H_{\mathrm{I}}^{\mathrm{tr}}] + \dots .$$

To first order we therefore have

$$H_{\mathrm{I}}^{\mathrm{tr}} \to \beta E_p + \tfrac{1}{2}g\lambda_0'[\lambda_0', H_1'^{\mathrm{e}}]_+ + \tfrac{1}{2}g\beta[\lambda_0', H_1'^{\mathrm{o}}]_+ + \dots . \tag{34}$$

3·2. *Transformation II.* – We can make a different simple choice for f in eq. (23), in order to obtain a c-number for the expression $f\lambda_0 + \lambda_0 f$. If we put

$$f = \lambda_0 , \qquad i.e. \qquad f\lambda_0 + \lambda_0 f = 2 ,$$

we get a two-step transformation

$$U_{\mathrm{II}} = \sqrt{\beta\lambda_0}\,\sqrt{\lambda_0\lambda} , \tag{35}$$

which transforms λ into β via the operater λ_0. The transition $\lambda_0 \to \beta$ is in fact a free particle F.W. transformation:

$$\sqrt{\beta\lambda_0} = U_0 = [2E_p(E_p + m)]^{-\frac{1}{2}}(E_p + \beta H_0) .$$

In the case of vanishing external fields we obviously have

$$U_{\mathrm{II}} \to \sqrt{\beta\lambda_0}\,\sqrt{\lambda_0\lambda_0} = \sqrt{\beta\lambda_0} = U_0 .$$

In the general (stationary) case we obtain the following expressions for the transformation function and the transformed Hamiltonian, to third order:

$$U_{\mathrm{II}} = U_0(1 + \tfrac{1}{2}g\lambda_0\lambda_1 + \tfrac{1}{8}g^2(3\lambda_0\lambda_2 + \lambda_1^2) + \tfrac{1}{16}g^3(6\lambda_0\lambda_3 - 2\lambda_3\lambda_0 + \lambda_0\lambda_1^3) + \dots) , \tag{36}$$

$$H_{\mathrm{II}}^{\mathrm{tr}} = \beta E_p + U_0(gH_1^{\mathrm{e}} + \tfrac{1}{4}g^2\lambda_0[\lambda_1, H_1^{\mathrm{a}}]_+ + \tfrac{1}{4}g^3\lambda_0[\lambda_2^{\mathrm{a}}, H_1^{\mathrm{a}}]_+ + \dots)U_0^{*} . \tag{37}$$

The index a (or e) means anticommuting (or commuting) part with respect to λ_0, as explained in eq. (7).

3·3. *Transformation III.* – This transformation is carried out in three steps,

$$\lambda \to \lambda_0 \to \eta \to \beta ,$$

that is

$$\left\{ \begin{array}{l} U_{\mathrm{III}} = \sqrt{\beta\eta}\,\sqrt{\eta\lambda_0}\,\sqrt{\lambda_0\lambda} , \\ \text{or} \\ U_{\mathrm{III}} = V_0\sqrt{\lambda_0\lambda} , \end{array} \right. \tag{38}$$

where V is the free particle transformation function which was found as the limiting case of U_{I}, eq. (27), *i.e.*

$$V_0 = \tfrac{1}{2}(1 + i\gamma_5)(1 + \eta\lambda_0) .$$

Due to the formal similarity of U_{III} and U_{II}, the following expansions can be written down by replacing U_0 with V_0 in eqs. (36) and (37) respectively:

$$U_{\mathrm{III}} = V_0(1 + \tfrac{1}{2}g\lambda_0\lambda_1 + \tfrac{1}{8}g^2(4\lambda_0\lambda_2 + \lambda_1^2) + \tfrac{1}{16}g^3(6\lambda_0\lambda_3 - 2\lambda_3\lambda_0 + \lambda_0\lambda_1^3) + \ldots) , \tag{39}$$

$$H_{\mathrm{III}}^{\mathrm{tr}} = \beta E_p + V_0(gH_1^{\mathrm{c}} + \tfrac{1}{4}g^2\lambda_0[\lambda_1, H_1^{\mathrm{a}}]_+ + \tfrac{1}{4}g^3\lambda_0[\lambda_2^{\mathrm{a}}, H_1^{\mathrm{a}}]_+ + \ldots)V_0^* . \tag{40}$$

In order to simplify calculations, we put U_{III} in a more practical form (as was done for U_{I}):

$$U_{\mathrm{III}} = \sqrt{\beta\eta}\sqrt{\eta\lambda_0}\sqrt{\lambda_0\lambda} = \sqrt{\beta\eta}\sqrt{\eta\lambda_0}\sqrt{\lambda_0\lambda}\sqrt{\eta\beta}\sqrt{\beta\eta} = \sqrt{\beta\lambda_0'}\sqrt{\lambda_0'\lambda'}\sqrt{\beta\eta} .$$

The dash denotes transformation with $\sqrt{\beta\eta}$, as explained in eq. (29). Then the transformed Hamiltonian can be written as

$$H_{\mathrm{III}}^{\mathrm{tr}} = \sqrt{\beta\lambda_0'}\sqrt{\lambda_0'\lambda'}H'\sqrt{\lambda'\lambda_0'}\sqrt{\lambda_0'\beta} , \qquad H' = \sqrt{\beta\eta}H\sqrt{\eta\beta} . \tag{41}$$

The transition from H to H' corresponds to a change in the Dirac-matrix representation, as mentioned earlier, and going from H' to H^{tr} is in fact transformation II applied to the Hamiltonian H'.

From (41)—and remembering (8)—we then get to first order in g

$$H_{\mathrm{III}}^{\mathrm{tr}} = \beta E_p + \tfrac{1}{2}g\sqrt{\beta\lambda_0'}\lambda_0'[\lambda_0', H_1']_+\sqrt{\lambda_0'\beta} , \tag{42}$$

where

$$\lambda_0' = (-\eta m + \boldsymbol{\alpha p})E_p^{-1} ,$$

$$H_1' = \tfrac{1}{2}(1 + i\gamma_5)H_1(1 - i\gamma_5) .$$

By means of the relation

$$\sqrt{\beta\lambda_0'} = \frac{1}{\sqrt{2}}(1 + \beta\lambda_0'),$$

we can now show that to first order $H_{\mathrm{III}}^{\mathrm{tr}}$ is equal to $H_{\mathrm{I}}^{\mathrm{tr}}$, given by (34):

$$\left\{\begin{aligned} H_{\mathrm{III}}^{\mathrm{tr}} &= \beta E_p + \tfrac{1}{4}g(1 + \beta\lambda_0')\lambda_0'[\lambda_0', H_1'](1 - \beta\lambda_0') \\ &= \beta E_p + \tfrac{1}{4}g(1 + \beta\lambda_0')\lambda_0'(1 + \beta\lambda_0')[\lambda_0', H_1'^{\mathrm{e}}]_+ + \\ &\qquad + \tfrac{1}{4}g(1 + \beta\lambda_0')\lambda_0'(1 - \beta\lambda_0')[\lambda_0', H_1'^{\mathrm{o}}]^+ \\ &= \beta E_p + \tfrac{1}{2}g\lambda_0'[\lambda_0', H_1'^{\mathrm{e}}]_+ + \tfrac{1}{2}g\beta[\lambda_0', H_1'^{\mathrm{o}}]_+ . \end{aligned}\right. \tag{43}$$

4. – Transformations by iteration.

In this section we shall proceed along the following lines:

a) By an iteration procedure, analogous to the F.W. method, the Hamiltonian can be transformed to an operator, THT^*, which commutes with λ_0 to any desired order in the coupling constant,

$$[THT^*, \lambda_0] = 0 . \tag{44}$$

b) Then a free particle transformation is made, which transforms λ_0 into β. This can be effected either by the operator U_0, eq. (28), or by the operator V_0, eq. (27):

$$U_0 \lambda_0 U_0^* = \beta , \qquad V_0 \lambda_0 V_0^* = \beta .$$

Then we obtain two different Hamiltonians, both of them being even

$$H_{\mathrm{II}'}^{\mathrm{tr}} = U_0 THT^* U_0^* , \tag{45}$$

$$H_{\mathrm{III}'}^{\mathrm{tr}} = V_0 THT^* V_0^* . \tag{46}$$

The notation used is due to the fact that these transformations are related to the transformations (35) and (38), respectively:

$$U_{\mathrm{II}} = U_0 \sqrt{\lambda_0 \lambda} , \qquad U_{\mathrm{III}} = V_0 \sqrt{\lambda_0 \lambda} .$$

The even character of the transformed Hamiltonians is easily demonstrated by

$$[H_{\mathrm{II}'}^{\mathrm{tr}}, \beta] = [U_0 THT^* U_0^*, \; U_0 \lambda_0 U_0^*] = U_0 [THT^*, \lambda_0] U_0^* = 0 ,$$

and quite analogously for $H_{\mathrm{III}'}^{\mathrm{tr}}$.

Iterative solution of the equation $[THT^*, \lambda_0] = 0$. In the present theory the role played by λ_0 is analogous to the role played by β in the F.W. theory. Instead of writing the Hamiltonian as the sum of an even and odd part, we now write it as the sum of a c-type part (commuting with λ_0) and an a-type part (anticommuting with λ_0):

$$H = H_0 + gH_1 = H_0 + gH_1^{\mathrm{c}} + gH_1^{\mathrm{a}} ,$$

$$H_1^{\mathrm{c}} = \tfrac{1}{2}(H_1 + \lambda_0 H_1 \lambda_0) ,$$

$$H_1^{\mathrm{a}} = \tfrac{1}{2}(H_1 - \lambda_0 H_1 \lambda_0) .$$

In order to remove the a-type terms of first order, we make a canonical transformation, generated by the operator gs_1, s_1 being an Hermitian a-type operator

$$[s_1, \lambda_0]_+ = 0 . \tag{47}$$

Then,

$$\begin{aligned} H \to H^{(1)} &= \exp[igs_1] H \exp[-igs_1] \\ &= H_0 + gH_1^c + gH_1^a + ig[s_1, H_0 + gH_1^c + gH_1^a] + \dots . \end{aligned}$$

We require the first order a-type part of this expression to vanish:

$$i[s_1, H_0] + H_1^a = 0 . \tag{48}$$

With the substitution

$$s_1 = \tfrac{1}{2} i \, x_1 \lambda_0 ,$$

we get the following equations for x_1:

$$[x_1, H_0] + [\lambda_0, H_1] = 0 , \qquad [x_1, \lambda_0]_+ = 0 .$$

These are just the same equations as those which defined λ_1 in a unique way:

$$[\lambda_1, H_0] + [\lambda_0, H_1] = 0 , \qquad [\lambda_1, \lambda_0]_+ = 0 .$$

Hence the only solution of (47) and (48) is

$$s_1 = \tfrac{1}{2} i \lambda_1 \lambda_0 . \tag{49}$$

To third order the transformed Hamiltonian generated by this operator is

$$\begin{aligned} H^{(1)} = H_0 + gH_1^c + \tfrac{1}{4} g^2 \lambda_0 [\lambda_1, H_1^a]_+ + \tfrac{1}{2} g^2 \lambda_0 [\lambda_1, H_1^c] - \\ - \tfrac{1}{8} g^3 \left[\lambda_1, [\lambda_1, H_1^c]\right] - \tfrac{1}{12} g^3 \left[\lambda_1, [\lambda_1, H_1^a]_+\right]_+ + \dots . \end{aligned} \tag{50}$$

The next transformation is one which removes the term $\frac{1}{2} g^2 \lambda_0 [\lambda_1, H_1^c]$. Requiring s_2 to be an a-type operator (corresponding to being odd in the F.W. theory), one can show that the only possible choice is to put

$$s_2 = \tfrac{1}{2} i \lambda_2^a \lambda_0 .$$

In general, the n'th generator s_n shall satisfy the equations

$$(51) \qquad \begin{cases} s_n = s_n^* , \\ [s_n, \lambda_0]_+ = 0 , \\ [s_n, H_0] = ih_n^a , \end{cases}$$

where h_n^a is some Hermitian a-type operator which depends on the preceding transformations. Multiplying the last equation from the right by λ_0, we get

$$[s_n, E_p]_+ = ih_n^a \lambda_0 ,$$

whose only solution, according to (11), is

$$s_n = \int_{-\infty}^{0} \exp[E_p \tau] ih_n^a \lambda_0 \exp[E_p \tau] \, d\tau .$$

Thus the iteration procedure outlined above can be carried out to any order, and the restriction, $[s_n, \lambda_0]_+ = 0$, makes the expansion unique.

We should like to point out a pleasant feature of the iteration method. Let us assume that n transformations have been carried out. Then the result, $H^{(n)}$, is a c-type operator to order n. By the next transformation, generated by the operator $g^{n+1}s_{n+1}$, the following expression is added to the Hamiltonian

$$H^{(n+1)} - H^{(n)} = ig^{n+1}[s_{n+1}, H^{(n)}] - \tfrac{1}{2}g^{2n+2}[s_{n+1}, [s_{n+1}, H^{(n)}]] + \ldots .$$

The lowest order c-type term in this expression is of order $2n+2$. That is, up to order $2n+1$ the c-type part of the Hamiltonian is not affected by this transformation. Obviously, this part is neither affected by the succeeding transformations. Hence, to order $2n+1$ the transformed Hamiltonian is equal to the c-type part of $H^{(n)}$. The transformed result to third order is therefore the c-type part of $H^{(1)}$, given by eq. (50),

$$THT^* = H_0 + gH_1^c + \tfrac{1}{4}g^2\lambda_0[\lambda_1, H_1^a]_+ - \tfrac{1}{8}g^3[\lambda_1, [\lambda_1, H_1^c]] + \ldots .$$

According to eq. (45) and (46) we then have the following even Hamiltonians

$$(52) \qquad H_{II'}^{tr} = \beta E_p + U_0(gH_1^c + \tfrac{1}{4}g^2\lambda_0[\lambda_1, H_1^a]_+ - \tfrac{1}{8}g^3[\lambda_1, [\lambda_1, H_1^c]] + \ldots)U_0^* ,$$

$$(53) \qquad H_{III'}^{tr} = \beta E_p + V_0(gH_1^c + \tfrac{1}{4}g^2\lambda_0[\lambda_1, H_1^a]_+ - \tfrac{1}{8}g^3[\lambda_1, [\lambda_1, H_1^c]] + \ldots)V_0^* ,$$

where

$$U_0 = [2E_p(E_p + m)]^{-\frac{1}{2}}(E_p + \beta H_0) \quad \text{and} \quad V_0 = \tfrac{1}{2}(1 + i\gamma_5)E_p^{-1}(E_p + \eta H_0) .$$

To second order in the coupling constant these results coincide with the results from transformations II and III respectively.

In contradiction to the transformations found in a closed form, (I, II and III), the iterative solutions are found without making use of the equation $U\lambda U^* = \beta$, which is the condition for the transformation to be energy separating. We shall show that this relation is valid for $U_{II'}$ and $U_{III'}$, as far as the expansions in powers of g are convergent. We make use of the property

$$[\lambda_0, THT^*] = 0\,, \quad \text{where } T = \ldots \exp[ig^2 s_2] \exp[igs_1]\,,$$

which is equivalent to

$$[T^*\lambda_0 T, H] = 0\,.$$

Obviously we also have

$$(T^*\lambda_0 T)^2 = 1 \quad \text{and} \quad (T^*\lambda_0 T)_{\text{zero order}} = \lambda_0\,.$$

These are however the same equations as (4), (5) and (6), which defined the expansion of λ in powers of g. Hence

$$T^*\lambda_0 T = \lambda\,,$$

or

$$T\lambda T^* = \lambda_0\,.$$

Making a unitary transformation of this relation, by means of U_0 and V_0 respectively, we get:

$$U_0 T\lambda T^* U_0^* = U_0\lambda_0 U_0^*\,, \qquad V_0 T\lambda T^* V_0^* = V_0\lambda_0 V_0^*\,,$$

i.e.

$$U_{II'}\,\lambda U_{II'}^* = \beta\,, \qquad U_{III'}\,\lambda U_{III'}^* = \beta\,.$$

B) Generalization to Time Dependent External Fields.

5. – Notation and introduction.

The Dirac equation will now be written as

(54) $$K\psi \equiv (K_0 + gK_1)\psi = 0\,,$$

where

(55) $$K_0 = H_0 - i\frac{\partial}{\partial t}\,, \qquad K_1 = H_1\,.$$

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Making the transformation $\psi^{tr} = U\psi$, the Hamiltonian is converted to

$$H^{tr} = UKU^* + i\frac{\partial}{\partial t}.$$

The necessary and sufficient condition for this expression to be an even operator is

$$[\beta, UKU^*] = 0,$$

i.e.

$$[U^*\beta U, K] = \left[U^*\beta U, H - i\frac{\partial}{\partial t}\right] = 0. \tag{56}$$

Hence, the operator $U^*\beta U$ has to be a constant of motion in the Dirac picture.

In the case of stationary external fields, the operator $U^*\beta U$ was put equal to $\lambda = H/\sqrt{H^2}$. This is, however, not possible in the present case, since in general

$$i\dot{\lambda} = \left[\lambda, H - i\frac{\partial}{\partial t}\right] = i\frac{\partial\lambda}{\partial t} \neq 0.$$

What we want is a generalization of λ to an operator Λ, with the following properties

$$i\dot{\Lambda} = [\Lambda, K] = \left[\Lambda, H - i\frac{\partial}{\partial t}\right] = 0, \tag{57}$$

$$\Lambda = \Lambda^* = \Lambda^{-1}. \tag{58}$$

Further, in the case of vanishing external fields we require

$$\Lambda \to \lambda_0 = H_0 E_p^{-1}, \qquad i.e. \quad \Lambda_0 = \lambda_0. \tag{59}$$

Then U can be found as a solution of the equation

$$U^*\beta U = \Lambda, \qquad i.e. \quad U\Lambda U^* = \beta, \tag{60}$$

which is quite analogous to eq. (20).

6. – Expansions of the operator Λ.

6·1. *Expansion in powers of g.* – We want to determine the operator coefficients Λ_i in the expansion

$$\Lambda = \Lambda_0 + g\Lambda_1 + g^2\Lambda_2 + \dots,$$

where

$$\Lambda_0 = \lambda_0 .$$

Requiring the equations

$$\Lambda^2 = 1 ,$$

$$\left[\Lambda, H_0 - i\frac{\partial}{\partial t} + gH_1\right] = 0 ,$$

to be satisfied to each order in g, we obtain the following equations,

$$\sum_{k=0}^{n} \Lambda_k \Lambda_{n-k} = 0 , \qquad n = 1, 2, \ldots, \tag{61}$$

$$\left[\Lambda_n, H_0 - i\frac{\partial}{\partial t}\right] + [\Lambda_{n-1}, H_1] = 0 , \qquad n = 1, 2, \ldots . \tag{62}$$

By means of (61) the c-type part of Λ_n (the part commuting with λ_0) can be expressed in terms of the Λ_i's of lower order:

$$\begin{aligned} \Lambda_1^c &= 0 , \\ \Lambda_2^c &= -\tfrac{1}{2}\Lambda_0 \Lambda_1^2 , \\ &\vdots \\ \Lambda_n^c &= -\tfrac{1}{2}\Lambda_0 \sum_{k=1}^{n-1} \Lambda_k \Lambda_{n-k} . \end{aligned}$$

Next, let us consider the a-type part of eq. (62):

$$\left[\Lambda_n^a, H_0 - i\frac{\partial}{\partial t}\right] = -\frac{1}{2}\Lambda_0 [\Lambda_0, [\Lambda_{n-1}, H_1]] .$$

Multiplication with Λ_0 gives the equation

$$[\Lambda_n^a, E_p]_+ - i\Lambda_0 \frac{\partial \Lambda_n^a}{\partial t} = B_n^a , \tag{63}$$

where B_n^a is the explicitly time dependent operator

$$B_n^a(t) = \tfrac{1}{2}[\Lambda_0, [\Lambda_{n-1}(t), H_1(t)]] .$$

The following solution has been found

$$\Lambda_n^a = \frac{1}{2}(1 + \Lambda_0) \int_{-\infty}^{0} \exp[E_p \tau] B_n^a(t - i\tau) \exp[E_p \tau] \, d\tau + \text{Herm. conj.} \tag{64}$$

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The proof goes as follows:

$$[\Lambda_n^a, E_p]_+ - i\Lambda_0 \frac{\partial \Lambda_n^a}{\partial t} =$$

$$= \tfrac{1}{2}(1+\Lambda_0)\int_{-\infty}^{0} \exp[E_p\tau]\{[B_n^a(t-i\tau), E_p]_+ - i\Lambda_0\,\partial B_n^a(t-i\tau)/\partial t\}\cdot$$

$$\cdot \exp[E_p\tau]\,\mathrm{d}\tau + \text{Herm. conj.} =$$

$$= \tfrac{1}{2}(1+\Lambda_0)\int_{-\infty}^{0} \exp[E_p\tau]\{E_p B_n^a(t-i\tau) + \partial B_n^a(t-i\tau)/\partial\tau + B_n^a(t-i\tau)E_p\}\cdot$$

$$\cdot \exp[E_p\tau]\,\mathrm{d}\tau + \text{Herm. conj.} =$$

$$= \tfrac{1}{2}(1+\Lambda_0)\int_{-\infty}^{0} \mathrm{d}\{\exp[E_p\tau]\,B_n^a(t-i\tau)\exp[E_p\tau]\} + \text{Herm. conj.} =$$

$$= \tfrac{1}{2}(1+\Lambda_0)\,B_n^a(t) + \tfrac{1}{2}\,B_n^a(t)(1+\Lambda_0) = B_n^a(t)\,.$$

When the operator B_n^a is explicitly independent of t, then this will also be true for the solution which we have found for Λ_n^a (*). In fact,

$$\Lambda_n^a \to \tfrac{1}{2}(1+\Lambda_0)\int_{-\infty}^{0} \exp[E_p\tau]\,B_n^a \exp[E_p\tau]\,\mathrm{d}\tau + \text{Herm. conj.} =$$

$$= \int_{-\infty}^{0} \exp[E_p\tau]\,B_n^a \exp[E_p\tau]\,\mathrm{d}\tau\,,$$

where

$$B_n^a = \tfrac{1}{2}[\Lambda_0, [\Lambda_{n-1}, H_1]]\,.$$

Comparing this expression with the previous result,

$$\lambda_n^a = \int_{-\infty}^{0} \exp[E_p\tau]\,b_n^a \exp[E_p\tau]\,\mathrm{d}\tau\,, \tag{11}$$

where

$$b_n^a = \tfrac{1}{2}[\lambda_0, [\lambda_{n-1}, H_1]]\,,$$

we see that for stationary external fields we obviously have $\Lambda = \lambda$.

(*) This property of Λ_n^a will be spoilt if one adds solutions of the homogenous equation.

In the general case the operator Λ_1 can be written as

$$\Lambda_1 = \tfrac{1}{2}(\lambda_0 - 1)\mathscr{I}_1^+ + \tfrac{1}{2}(\lambda_0 + 1)\mathscr{I}_1^- ,$$

where

$$\mathscr{I}_1^{\pm} = \int_{-\infty}^{0} \exp[E_p\tau][\lambda_0, H_1(t \pm i\tau)] \exp[E_p\tau]\,\mathrm{d}\tau .$$

We shall give two examples:

a) $H_1 = -\boldsymbol{E}\boldsymbol{r} f(t)$, where $\boldsymbol{E}$ is a constant and $f(t)$ a *c*-number function.

$$\mathscr{I}_1^{\pm} = \int_{-\infty}^{0} \exp[E_p\tau][\lambda_0, -\boldsymbol{E}\boldsymbol{r}] f(t \pm i\tau) \exp[E_p\tau]\,\mathrm{d}\tau =$$

$$= \int_{-\infty}^{0} \exp[E_p\tau]\, i\boldsymbol{E} \frac{\partial \lambda_0}{\partial \boldsymbol{p}} f(t \pm i\tau) \exp[E_p\tau]\,\mathrm{d}\tau =$$

$$= i\boldsymbol{E} \frac{\partial \lambda_0}{\partial \boldsymbol{p}} \int_{-\infty}^{0} f(t \pm i\tau) \exp[2E_p\tau]\,\mathrm{d}\tau .$$

Putting for instance $f(t) = \exp[i\omega t]$, we get

$$\mathscr{I}_1^{\pm} = i \exp[i\omega t]\, \boldsymbol{E} \frac{\partial \lambda_0}{\partial \boldsymbol{p}} (2E_p \mp \omega)^{-1} ,$$

$$\Lambda_1 = \exp[i\omega t] E_p^{-1} (4E_p^2 - \omega^2)^{-1} \left(2im\beta\boldsymbol{\alpha}\boldsymbol{E} - 2\boldsymbol{\sigma}(\boldsymbol{p} \times \boldsymbol{E}) + i\omega\lambda_0 E_p^{-1} \boldsymbol{E}\boldsymbol{p} - i\omega\boldsymbol{\alpha}\boldsymbol{E}\right) .$$

When $\omega \to 0$,

$$\Lambda_1 \to \lambda_1 = \tfrac{1}{2} E_p^{-3} \left(im\beta\boldsymbol{\alpha}\boldsymbol{E} - \boldsymbol{\sigma}(\boldsymbol{p} \times \boldsymbol{E})\right) ,$$

which is the previous result, eq. (14).

b) We consider a Fourier component $H_1(\boldsymbol{k}) = \exp[i\boldsymbol{k}\boldsymbol{r} - i\omega t]$. By means of previous calculations, leading to eq. (15), we have

$$\exp[E_p\tau][\lambda_0, \exp[i\boldsymbol{k}\boldsymbol{r}]] = \exp[i\boldsymbol{k}\boldsymbol{r}](\lambda_0' - \lambda_0) \exp[E_p'\tau] ,$$

where

$$E_p' = [m^2 + (\boldsymbol{p} + \boldsymbol{k})^2]^{\frac{1}{2}} , \qquad \lambda_0' = (H_0 + \boldsymbol{\alpha}\boldsymbol{k})/E_p' .$$

10:0

Hence

$$\mathscr{I}_1^{\pm} = \int_{-\infty}^{0} \exp[E_p\tau]\,[\lambda_0,\, \exp[i\boldsymbol{kr}]]\exp[-i\omega t \pm \omega\tau]\exp[E_p\tau]\,\mathrm{d}\tau =$$

$$= \exp[i\boldsymbol{kr} - i\omega t](\lambda_0' - \lambda_0)\int_{-\infty}^{0}\exp[(E_p' + E_p \pm \omega)\tau]\,\mathrm{d}\tau =$$

$$= \exp[i\boldsymbol{kr} - i\omega t](\lambda_0' - \lambda_0)(E_p' + E_p \pm \omega)^{-1}\,,$$

$$\Lambda_1 = \tfrac{1}{2}\exp[i\boldsymbol{kr} - i\omega t]\,\{(1-\lambda_0')(E_p' + E_p + \omega)^{-1} + (1+\lambda_0')(E_p' + E_p - \omega)^{-1}\}(1-\lambda_0'\lambda_0)\,.$$

When $\omega \to 0$,

$$\Lambda_1' \to \lambda_1 = \exp[i\boldsymbol{kr}](1-\lambda_0'\lambda_0)(E_p' + E_p)^{-1}\,,$$

which is the previous result, eq. (15).

6·2. *Expansion in powers of* $1/m$. – We want to find the coefficients $\Lambda_{(n)}$ in the expansion,

$$\Lambda = \Lambda_{(0)} + (1/m)\,\Lambda_{(1)} + \ldots,$$

when Λ is an operator satisfying the equations

$$\text{(65)} \qquad \begin{cases} \Lambda^2 = 1\,, \\ \left[\Lambda,\, H - i\dfrac{\partial}{\partial t}\right] = 0\,, \\ \Lambda_{(0)} = \beta\,, \end{cases}$$

$$H = m\beta + h = m\beta + \mathscr{E} + \mathscr{O}\,.$$

The commutator equation can be written as

$$[\Lambda,\, m\beta + h'] = [\Lambda,\, m\beta + \mathscr{E}' + \mathscr{O}] = 0\,,$$

where

$$\mathscr{E}' = \mathscr{E} - i\frac{\partial}{\partial t}\,.$$

Then (65) is formally equivalent to:

$$\text{(16)} \qquad \lambda^2 = 1\,,$$

$$\text{(17)} \qquad [\lambda,\, m\beta + \mathscr{E} + \mathscr{O}] = 0\,,$$

$$\text{(18)} \qquad \lambda_{(0)} = \beta\,.$$

Hence we can make use of the previous result (19):

$$\Lambda_{(n)} = \tfrac{1}{2}\beta\Big([\Lambda_{(n-1)}, h^{\prime}] - \sum_{k=1}^{n-1} \Lambda_{(k)} \Lambda_{(n-k)}\Big) .$$

Since the operator $\partial/\partial t$ will always appear in commutators, the operators $\Lambda_{(n)}$ will not contain differential operators in t.

As a curiosity we would like to mention that the operator,

$$K/\sqrt{K^2} = \left(H - i\frac{\partial}{\partial t}\right)\Big/\sqrt{\left(H - i\frac{\partial}{\partial t}\right)^2} ,$$

where $i(\partial/\partial t)$ is being considered as being of zero order, gives an expansion in powers of $1/m$ which is just the one we have found.

7. – Transformations in closed form.

As has been shown before, eq. (60), U can be found as a solution of the equation

(60) $$U^*\beta U = \Lambda$$

which is quite analogous to

(20) $$U^*\beta U = \lambda .$$

Obviously, the previous transformations can be generalized to

$$\begin{aligned} U_{\mathrm{E}} &= \sqrt{\beta\Lambda} , \\ U_{\mathrm{I}} &= \sqrt{\beta\eta}\,\sqrt{\eta\Lambda} , \\ U_{\mathrm{II}} &= \sqrt{\beta\Lambda_0}\,\sqrt{\Lambda_0\Lambda} , \\ U_{\mathrm{III}} &= \sqrt{\beta\eta}\,\sqrt{\eta\Lambda_0}\,\sqrt{\Lambda_0\Lambda} . \end{aligned}$$

Introducing the expansion

$$\Lambda = \Lambda_0 + g\Lambda_1 + \dots ,$$

the transformations I, II and III can be expanded in powers of g. The results are quite analogous to the previous ones, given by eq. (26), (36) and (39), respectively. The only modification being necessary is to replace λ_i by Λ_i.

For the transformed Hamiltonian we find:

$$(66') \quad H_{\mathrm{I}}^{\mathrm{tr}} = \beta E_p + \tfrac{1}{2} g \left(\Lambda_0'[\Lambda_0', H_1'^{\mathrm{e}}]_+ + \beta[\Lambda_0', H_1'^{\mathrm{o}}]_+ + \beta[\Lambda_1'^{\mathrm{o}}, H_0']_+ - i\Lambda_0' \frac{\partial \Lambda_1'^{\mathrm{o}}}{\partial t} \right) + \dots ,$$

$$(67) \quad H_{\mathrm{II}}^{\mathrm{tr}} = \beta E_p + U_0 (g H_1^{\mathrm{c}} + \tfrac{1}{4} g^2 \Lambda_0[\Lambda_1, H_1^{\mathrm{a}}]_+ + \tfrac{1}{4} g^3 \Lambda_0[\Lambda_2^{\mathrm{a}}, H_1^{\mathrm{a}}]_+ + \dots) U_0^* ,$$

$$(68) \quad H_{\mathrm{III}}^{\mathrm{tr}} = \beta E_p + V_0 (g H_1^{\mathrm{c}} + \tfrac{1}{4} g^2 \Lambda_0[\Lambda_1, H_1^{\mathrm{a}}]_+ + \tfrac{1}{4} g^3 \Lambda_0[\Lambda_2^{\mathrm{a}}, H_1^{\mathrm{a}}]_+ + \dots) V_0^* ,$$

where as usual the dash denotes the transformation $F' = \sqrt{\eta\beta} F \sqrt{\beta\eta}$.

By means of an additional even transformation of $H_{\mathrm{I}}^{\mathrm{tr}}$, generated by the operator $S_1 = (i/2) g \Lambda_1'^{\mathrm{o}} \Lambda_0'$, the terms containing $\Lambda_1'^{\mathrm{o}}$ can be removed to first order:

$$\exp[iS_1] \left(H_{\mathrm{I}}^{\mathrm{tr}} - i \frac{\partial}{\partial t} \right) \exp[-iS_1] = H_{\mathrm{I}}^{\mathrm{tr}} - i \frac{\partial}{\partial t} - \frac{1}{2} g \left[\Lambda_1'^{\mathrm{o}} \Lambda_0', \beta E_p - i \frac{\partial}{\partial t} \right] =$$

$$= H_{\mathrm{I}}^{\mathrm{tr}} - i \frac{\partial}{\partial t} - \frac{i}{2} g \frac{\partial \Lambda_1'^{\mathrm{o}}}{\partial t} \Lambda_0' - \frac{1}{2} g [\Lambda_1'^{\mathrm{o}} \Lambda \quad \beta \Lambda_0' H_0'] + \dots =$$

$$= H_{\mathrm{I}}^{\mathrm{tr}} - i \frac{\partial}{\partial t} + \frac{i}{2} g \Lambda_0' \frac{\partial \Lambda_1'^{\mathrm{o}}}{\partial t} - \frac{1}{2} g \beta [\Lambda_1'^{\mathrm{o}}, H_0']_+ + \dots .$$

That is,

$$(66) \qquad H_{\mathrm{I}}^{\mathrm{tr}} \to \beta E_p + \tfrac{1}{2} g \lambda_0'[\lambda_0', H_1'^{\mathrm{e}}]_+ + \tfrac{1}{2} g \beta [\lambda_0', H_1'^{\mathrm{o}}]_+ + \dots .$$

The results, given by (66), (67) and (68), are to first order in g equal to corresponding transformation results for time-independent external fields, given by eq. (34), (37) and (40), respectively.

8. – Transformations by iteration.

We shall briefly discuss how the iteration solution in powers of g can be modified in the case of time-dependent external fields.

The Dirac equation is written as

$$K\psi \equiv (K_0 + g K_1^{\mathrm{c}} + g K_1^{\mathrm{a}}) \psi = 0 ,$$

where

$$K_0 = H_0 - i \frac{\partial}{\partial t} , \qquad K_1^{\mathrm{c}} = H_1^{\mathrm{c}} , \qquad K_1^{\mathrm{a}} = H_1^{\mathrm{a}} .$$

By successive transformations we want to determine a total transformation T:

$$\psi \to T\psi ,$$
$$H \to THT^* - iT \frac{\partial T^*}{\partial t} = TKT^* + i \frac{\partial}{\partial t} ,$$

such that the Hamiltonian is converted to a c-type operator

$$\left[TKT^* + i\frac{\partial}{\partial t}, \Lambda_0\right] = 0\,,$$

i.e.

$$[TKT^*, \Lambda_0] = 0\,.$$

In order to remove the a-type terms of first order in K, we make a transformation generated by an operator gS_1,

$$S_1 = S_1^*\,, \qquad [S_1, \Lambda_0]_+ = 0\,.$$

Then,

$$K \to K^{(1)} = \exp[igS_1]K\exp[-igS_1] =$$
$$= K_0 + gK_1^c + gK_1^a + ig[S_1, K_0 + gK_1^c + gK_1^a] + \ldots\,.$$

Requiring

$$i[S_1, K_0] + K_1^a = 0\,,$$

and making the substitution

$$S_1 = \frac{i}{2}X_1\Lambda_0\,,$$

we get the following equations,

$$\left[X_1, H_0 - i\frac{\partial}{\partial t}\right] + [\Lambda_0, H_1] = 0\,, \qquad [X_1, \Lambda_0]_+ = 0\,,$$

which are satisfied by $X_1 = \Lambda_1$. Hence

$$S_1 = \frac{i}{2}\Lambda_1\Lambda_0\,.$$

The corresponding operator in the time independent case was

(49) $$s_1 = \frac{i}{2}\lambda_1\lambda_0\,.$$

To third order the operator S_1 generates the result:

$$K \to K^{(1)} = K_0 + gK_1^c + \tfrac{1}{4}g^2\Lambda_0[\Lambda_1, K_1^a]_+ + \tfrac{1}{2}g^2\Lambda_0[\Lambda_1, K_1^c] -$$
$$- \tfrac{1}{8}g^3[\Lambda_1, [\Lambda_1, K_1^c]] - \tfrac{1}{12}g^3[\Lambda_1, [\Lambda_1, K_1^a]\,]_+ + \ldots\,.$$

The next transformation is generated by the operator $(i/2)g^2\Lambda_2^a\Lambda_0$, which removes the a-type term $\frac{1}{2}g^2\Lambda_0[\Lambda_1, K_1^c]$.

According to a previous statement, the transformed K to third order in g is equal to the c-type part of $K^{(1)}$:

$$TKT^* = K_0 + gK_1^c + \tfrac{1}{4}g^2\Lambda_0[\Lambda_1, K_1^a]_+ - \tfrac{1}{8}g^3\left[\Lambda_1, [\Lambda_1, K_1^c]\right] + \dots ;$$

$$H \to THT^* - i\,T\frac{\partial T^*}{\partial t} = TKT^* + i\frac{\partial}{\partial t} =$$

$$= H_0 + gH_1^c + \tfrac{1}{4}g^2\Lambda_0[\Lambda_1, H_1^a]_+ - \tfrac{1}{8}g^3\left[\Lambda_1, [\Lambda_1, H_1^c]\right] + \dots .$$

This Hamiltonian, which commutes with λ_0, can now be transformed by either of the free particle transformations, U_0 or V_0, to an even operator:

$$(69)\qquad H_{\mathrm{II}'}^{\mathrm{tr}} = \beta E_p + U_0(gH_1^c + \tfrac{1}{4}g^2\Lambda_0[\Lambda_1, H_1^a]_+ - \tfrac{1}{8}g^3\left[\Lambda_1, [\Lambda_1, H_1^c]\right] + \dots)U_0^* ,$$

$$(70)\qquad H_{\mathrm{III}'}^{\mathrm{tr}} = \beta E_p + V_0(gH_1^c + \tfrac{1}{4}g^2\Lambda_0[\Lambda_1, H_1^a]_+ - \tfrac{1}{8}g^3\left[\Lambda_1, [\Lambda_1, H_1^c]\right] + \dots)V_0^* .$$

To first order in g, these results are exactly equal to those which were found by iteration methods in the case of time-independent external fields, eq. (52) and (53).

C) Electromagnetic External Fields.

9. – Dirac particle in electromagnetic field.

9·1. *Transformed Hamiltonians to first order in e.* – We consider the Hamiltonian for a Dirac particle with electric charge e in a general time-dependent electromagnetic field,

$$H = \beta m + \boldsymbol{\alpha}\boldsymbol{p} + e(\Phi - \boldsymbol{\alpha}\boldsymbol{A}) = H_0 + eH_1 .$$

According to eq. (67) and (69), the transformations II and II′ give the same Hamiltonian to first order in e:

$$H_{\mathrm{II}}^{\mathrm{tr}} = \beta E_p + eU_0H_1^cU_0^* , \qquad U_0 = \sqrt{\beta\lambda_0} = \left[2E_p(E_p + m)\right]^{-\frac{1}{2}}(E_p + \beta H_0) .$$

By means of the relations, $U_0\lambda_0 = \beta U_0$ and $\lambda_0 U_0^* = U_0^*\beta$, we have,

$$U_0H_1^cU_0^* = \tfrac{1}{2}U_0(H_1 + \lambda_0H_1\lambda_0)U_0^* = \tfrac{1}{2}U_0H_1U_0^* + \tfrac{1}{2}\beta U_0H_1U_0^*\beta = (U_0H_1U_0^*)^e =$$

$$= (e/2)\left[E_p(E_p + m)\right]^{-\frac{1}{2}}\{(E_p + m)\Phi(E_p + m) + (\boldsymbol{\alpha}\boldsymbol{p})\Phi(\boldsymbol{\alpha}\boldsymbol{p}) -$$

$$- \beta(\boldsymbol{\alpha}\boldsymbol{p})(\boldsymbol{\alpha}\boldsymbol{A})(E_p + m) - (E_p + m)(\boldsymbol{\alpha}\boldsymbol{A})(\boldsymbol{\alpha}\boldsymbol{p})\beta\}\left[E_p(E_p + m)\right]^{-\frac{1}{2}} .$$

This gives the following Hamiltonian:

$$(71)\qquad H_{\mathrm{II}}^{\mathrm{tr}} = \beta E_p + \frac{e}{2}\sqrt{1+\frac{m}{E_p}}\left\{\Phi + \frac{\boldsymbol{p}}{E_p+m}\Phi\frac{\boldsymbol{p}}{E_p+m} - \right.$$

$$- \boldsymbol{S}\frac{1}{E_p+m}(\boldsymbol{F}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{F})\frac{1}{E_p+m} - \beta\boldsymbol{S}\left(\frac{1}{E_p+m}\nabla\times\boldsymbol{A}+\nabla\times\boldsymbol{A}\,\frac{1}{E_p+m}\right) -$$

$$\left. - \beta\left(\frac{\boldsymbol{p}}{E_p+m}\boldsymbol{A}+\boldsymbol{A}\frac{\boldsymbol{p}}{E_p+m}\right)\right\}\sqrt{1+\frac{m}{E_p}}\,,$$

where

$$\boldsymbol{S} = \tfrac{1}{2}\boldsymbol{\sigma} \qquad \text{and} \qquad \boldsymbol{F} = -\nabla\Phi + i[\beta E_p, \boldsymbol{A}]\,.$$

According to eq. (68) and (70), the transformations III and III′ also give equal Hamiltonians to first order in e:

$$H_{\mathrm{III}}^{\mathrm{tr}} = \beta E_p + eV_0 H_1^{\mathrm{e}} V_0^*\,.$$

As was demonstrated in eq. (43), this can be expressed as

$$H_{\mathrm{III}}^{\mathrm{tr}} = \beta E_p + \tfrac{1}{2}e\lambda_0'[\lambda_0', H_1'^{\mathrm{e}}]_+ + \tfrac{1}{2}e\beta[\lambda_0', H_1'^{\mathrm{o}}]_+\,,$$

which is equal to the Hamiltonian H_1^{tr} given by eq. (34), when this has been modified by an additional transformation. Introducing

$$H_1' = \tfrac{1}{2}(1+i\gamma_5)(\Phi - \boldsymbol{\alpha}\boldsymbol{A})(1-i\gamma_5) = \Phi - \boldsymbol{\alpha}\boldsymbol{A}\,,$$

we easily find:

$$(72)\quad H_{\mathrm{I,III}}^{\mathrm{tr}} = \beta E_p + \tfrac{1}{2}e(\Phi + \lambda_0'\Phi\lambda_0') - \tfrac{1}{2}e\beta(\lambda_0'\boldsymbol{\alpha}\boldsymbol{A} + \boldsymbol{\alpha}\boldsymbol{A}\,\lambda_0') =$$

$$= \beta E_p + \frac{e}{2E_p}\{E_p\Phi E_p + (\boldsymbol{\sigma}\boldsymbol{p}+i\beta m)\Phi(\boldsymbol{\sigma}\boldsymbol{p}-i\beta m) -$$

$$- (\beta\boldsymbol{\sigma}\boldsymbol{p}+im)\boldsymbol{\sigma}\boldsymbol{A}\,E_p - E_p\boldsymbol{\sigma}\boldsymbol{A}(\beta\boldsymbol{\sigma}\boldsymbol{p}-im)\}\frac{1}{E_p} =$$

$$= \beta E_p + \frac{e}{2}\left(\Phi + \frac{m}{E_p}\Phi\frac{m}{E_p} + \frac{\boldsymbol{p}}{E_p}\Phi\frac{\boldsymbol{p}}{E_p}\right) - \boldsymbol{S}\frac{e}{2E_p}(\boldsymbol{F}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{F})\frac{1}{E_p} +$$

$$+ \beta\boldsymbol{S}\frac{me}{E_p}\boldsymbol{F}\frac{1}{E_p} - \beta\boldsymbol{S}\frac{e}{2}\left(\frac{1}{E_p}\nabla\times\boldsymbol{A}+\nabla\times\boldsymbol{A}\,\frac{1}{E_p}\right) - \beta\frac{e}{2}\left(\frac{\boldsymbol{p}}{E_p}\boldsymbol{A}+\boldsymbol{A}\frac{\boldsymbol{p}}{E_p}\right),$$

where

$$\boldsymbol{S} = \tfrac{1}{2}\boldsymbol{\sigma}\,, \qquad \text{and} \qquad \boldsymbol{F} = -\nabla\Phi + i[\beta E_p, \boldsymbol{A}]\,.$$

We now want to show that the quantity $i[\beta E_p, \boldsymbol{A}]$ in $\boldsymbol{F}$ can be replaced by $-\partial \boldsymbol{A}/\partial t$ by means of simple, even, unitary transformations. Then the operator $\boldsymbol{F}$ is replaced by the total electric field, $\boldsymbol{E} = -\nabla\Phi - \partial \boldsymbol{A}/\partial t$. In general we put

$$H^{\mathrm{Tr}} = \exp[ies]\left(H^{\mathrm{tr}} - i\frac{\partial}{\partial t}\right)\exp[-ies] + i\frac{\partial}{\partial t} = H^{\mathrm{tr}} + ie\left[s, H^{\mathrm{tr}} - i\frac{\partial}{\partial t}\right] + \dots,$$

where s is some zero order even and Hermitian operator. The first order equation is

$$(H^{\mathrm{Tr}})_1 = (H^{\mathrm{tr}})_1 + i[s, \beta E_p] - \frac{\partial s}{\partial t}.$$

Hence, if some part of $(H^{\mathrm{tr}})_1$ can be written as $i[\beta E_p, s]$, then this part will be replaced by $-\partial s/\partial t$. We easily see that we can choose

$$s = -\boldsymbol{S}\frac{1}{E_p + m}(\boldsymbol{A}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{A})\frac{1}{E_p + m} \qquad \text{in eq. (71)}$$

and

$$s = -\boldsymbol{S}\frac{1}{2E_p}(\boldsymbol{A}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{A} + 2m\beta\boldsymbol{A})\frac{1}{E_p} \qquad \text{in eq. (72)}$$

After these additional transformations we now get the following Hamiltonians for a fermion in a general time-dependent electromagnetic field, to first order in e:

(73)
$$H^{\mathrm{Tr}}_{\mathrm{II}} = \beta E_p + \frac{e}{2}\sqrt{1 + \frac{m}{E_p}}\left\{\Phi + \frac{\boldsymbol{p}}{E_p + m}\Phi\frac{\boldsymbol{p}}{E_p + m} - \right.$$
$$- \boldsymbol{S}\frac{1}{E_p + m}(\boldsymbol{E}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{E})\frac{1}{E_p + m} - \beta\boldsymbol{S}\left(\frac{1}{E_p + m}\boldsymbol{H} + \boldsymbol{H}\frac{1}{E_p + m}\right) -$$
$$\left. - \beta\left(\frac{\boldsymbol{p}}{E_p + m}\boldsymbol{A} + \boldsymbol{A}\frac{\boldsymbol{p}}{E_p + m}\right)\right\}\sqrt{1 + \frac{m}{E_p}},$$

(74)
$$H^{\mathrm{Tr}}_{\mathrm{I,III}} = \beta E_p + \frac{e}{2}\left(\Phi + \frac{m}{E_p}\Phi\frac{m}{E_p} + \frac{\boldsymbol{p}}{E_p}\Phi\frac{\boldsymbol{p}}{E_p}\right) -$$
$$- \boldsymbol{S}\frac{e}{2E_p}(\boldsymbol{E}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{E})\frac{1}{E_p} + \beta\boldsymbol{S}\frac{me}{E_p}\boldsymbol{E}\frac{1}{E_p} -$$
$$- \beta\boldsymbol{S}\frac{e}{2}\left(\frac{1}{E_p}\boldsymbol{H} + \boldsymbol{H}\frac{1}{E_p}\right) - \beta\frac{e}{2}\left(\frac{\boldsymbol{p}}{E_p}\boldsymbol{A} + \boldsymbol{A}\frac{\boldsymbol{p}}{E_p}\right).$$

At last we remember the simple result, eq. (32), of transformation I, when

H_1' is even and time-independent:

$$H_{\mathrm{I}}^{\mathrm{Tr}} = U_{\mathrm{I}} H U_{\mathrm{I}}^{*} = \beta E_p + \tfrac{1}{2} g(H_1' + \lambda_0' H_1' \lambda_0') + \dots ,$$

e.g. for a fermion in an electrostatic field, $H_1' = \Phi' = \Phi$:

$$H_{\mathrm{I}}^{\mathrm{Tr}} = \beta E_p + \frac{e}{2}\left(\Phi + \frac{m}{E_p}\Phi\frac{m}{E_p} + \frac{\boldsymbol{p}}{E_p}\Phi\frac{\boldsymbol{p}}{E_p}\right) + \boldsymbol{S}\frac{e}{E_p}(\nabla\Phi\times\boldsymbol{p})\frac{1}{E_p} - \beta\boldsymbol{S}\frac{me}{E_p}\nabla\Phi\frac{1}{E_p} .$$

9·2. *Low and high energy limits.* – At low energies ($p \ll m$) and to second order in $1/m$ we get

$$H_{\mathrm{II}}^{\mathrm{Tr}} \simeq \beta m + \frac{p^2}{2m} + e\Phi - \frac{e}{2m^2}\boldsymbol{S}\frac{1}{2}(\boldsymbol{E}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{E}) + \frac{e}{8m^2}\nabla^2\Phi - \tag{75}$$
$$- \beta\frac{e}{m}\boldsymbol{S}\boldsymbol{H} - \beta\frac{e}{2m}(\boldsymbol{p}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{p}) ,$$

$$H_{\mathrm{I,III}}^{\mathrm{Tr}} \simeq \beta m + \frac{p^2}{2m} + e\Phi - \frac{e}{m^2}\boldsymbol{S}\frac{1}{2}(\boldsymbol{E}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{E}) + \frac{e}{4m^2}\nabla^2\Phi + \tag{76}$$
$$+ \beta\frac{e}{m}\boldsymbol{S}\boldsymbol{E} - \beta\frac{e}{m}\boldsymbol{S}\boldsymbol{H} - \beta\frac{e}{2m}(\boldsymbol{p}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{p}) .$$

To the order considered, expression (75) coincides with Foldy and Wouthuysen's result. Expression (76), however, contains twice the spin orbit term, *i.e.* without the Thomas-Frenkel correction. On the other hand there appears a static coupling to the electric field, $\beta(e/m)\boldsymbol{S}\boldsymbol{E}$. If this term is considered individually it suggests an electric dipole moment (-operator) equal to $-\beta(e/m)\boldsymbol{S}$, *i.e.* equal to the magnetic moment, but of opposite sign. The so-called Darwin terms, $(e/8m^2)\nabla^2\Phi$ and $(e/4m^2)\nabla^2\Phi$, are to be discussed later, in connection with the charge distribution.

At high energies ($p \gg m$) all Hamiltonians contain the same leading terms to first order in e:

$$H_{\mathrm{I,II,III}}^{\mathrm{Tr}} \simeq \beta p + \frac{e}{2}\left(\Phi + \frac{\boldsymbol{p}}{p}\Phi\frac{\boldsymbol{p}}{p}\right) - \frac{e}{2}\beta\left(\frac{\boldsymbol{p}}{p}\boldsymbol{A} + \boldsymbol{A}\frac{\boldsymbol{p}}{p}\right) . \tag{77}$$

The characteristic feature is the dependence of the scalar potential on the direction of the momentum.

9·3. *Transition to the classical case.* – This is accomplished by making all dynamical variables commute. Even though that corresponds to putting $\hbar = 0$, we keep the spin variable $\boldsymbol{S}$, just in order to see how it enters into the

classical expressions. We also put $\beta = 1$ for positive energies. From eq. (73) and (74) we then obtain the following classical Hamiltonians, to first order in e:

$$H_{II}^{cl} = E_p + e\Phi - \frac{e}{E_p(E_p+m)} \boldsymbol{S}(\boldsymbol{E}\times\boldsymbol{p}) - \frac{e}{E_p}\boldsymbol{SH} - \frac{e}{E_p}\boldsymbol{Ap}\,,$$

$$H_{I,III}^{cl} = E_p + e\Phi - \frac{e}{E_p^2}\boldsymbol{S}(\boldsymbol{E}\times\boldsymbol{p}) + e\,\frac{m}{E_p^2}\boldsymbol{SE} - \frac{e}{E_p}\boldsymbol{SH} - \frac{e}{E_p}\boldsymbol{Ap}\,.$$

We will also express the energy, being equal to the Hamiltonian, by the particle velocity $\boldsymbol{v}$. Introducing conventional units, and dropping the term $e\boldsymbol{A}$ in $\boldsymbol{p}$, (second order!), we get to first order in e

$$E_{II}^{cl} = \frac{mc^2}{\gamma} + e\Phi - \frac{\gamma}{1+\gamma}\frac{e}{mc}\boldsymbol{S}\left(\frac{1}{c}\boldsymbol{E}\times\boldsymbol{v}\right) - \gamma\,\frac{e}{mc}\boldsymbol{SH} - \frac{e}{c}\boldsymbol{vA}\,,$$

$$E_{I,III}^{cl} = \frac{mc^2}{\gamma} + e\Phi - \gamma\,\frac{e}{mc}\boldsymbol{S}\left(\frac{1}{c}\boldsymbol{E}\times\boldsymbol{v}\right) + \gamma^2\,\frac{e}{mc}\boldsymbol{SE} - \gamma\,\frac{e}{mc}\boldsymbol{SH} - \frac{e}{c}\boldsymbol{vA}\,,$$

where

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}\,.$$

9·4. *Born approximation in one of the new representations.* – As has been pointed out, the main feature of the present transformations is that the wave equation splits in 2 two-component equations, corresponding to positive and negative energies. Let us now consider a Dirac particle with positive energy scattered by an electrostatic field Φ. Choosing transformation II, we get from (73) the wave equation

$$i\,\frac{\partial\varphi}{\partial t} = E_p\varphi + \frac{e}{2}\sqrt{1+\frac{m}{E_p}}\cdot$$
$$\cdot\left(\Phi + \frac{\boldsymbol{p}}{E_p+m}\Phi\frac{\boldsymbol{p}}{E_p+m} + \boldsymbol{S}\,\frac{2}{E_p+m}\,\boldsymbol{p}\times\boldsymbol{E}\,\frac{1}{E_p+m}\right)\sqrt{1+\frac{m}{E_p}}\,\varphi + \ldots\,,$$

where φ is the two-component wave function.

As can easily be verified, this can also be written as

$$(78)\qquad i\,\frac{\partial\varphi}{\partial t} = E_p\varphi + \frac{e}{2}\sqrt{1+\frac{m}{E_p}}\left(\Phi + \frac{\boldsymbol{\sigma p}}{E_p+m}\Phi\frac{\boldsymbol{\sigma p}}{E_p+m}\right)\sqrt{1+\frac{m}{E_p}}\,\varphi + \ldots\,,$$

which form is convenient for the calculations below. We introduce free par-

ticle states which are eigenstates of E_p, $\boldsymbol{p}$ and $\sigma_p \equiv \boldsymbol{\sigma p}/p$:

$$|E, \boldsymbol{p}', +\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \exp[i\boldsymbol{p}'\boldsymbol{r}], \qquad \sigma'_p = 1,$$

$$|E, \boldsymbol{p}', -\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \exp[i\boldsymbol{p}'\boldsymbol{r}], \quad \sigma'_p = -1.$$

$E = (m^2 + p'^2)^{\frac{1}{2}}$; θ is the angle between $\boldsymbol{p}'$ and the z-axis (in the xz-plane). It should be noted that to zero order in e: $\boldsymbol{p}^{\mathrm{Tr}} = \boldsymbol{p}$ and $\sigma_p^{\mathrm{Tr}} = \sigma_p$.

These states are simpler and more « natural » than in Dirac's representation. On the other hand the transformed Hamiltonians are in general far more complicated, but seem—in return—to lend themselves more easily to physical interpretations than the mathematically simpler Dirac Hamiltonian does.

Let us calculate the matrix element of $(H_{\mathrm{II}}^{\mathrm{Tr}})_1$, as given by eq. (78), between an initial state with momentum $\boldsymbol{p}_i = p\boldsymbol{k}$ along the z-axis, and a final state with momentum $\boldsymbol{p}_f = p(\boldsymbol{k}\cos\theta + \boldsymbol{i}\sin\theta)$ in the xz-plane.

As E_p and $\boldsymbol{\sigma p}$ are quantized in both cases, we obtain

$$\langle E, \boldsymbol{p}_f, + | H_1^{\mathrm{Tr}} | E, \boldsymbol{p}_i, + \rangle = \frac{1}{2}\left(1 + \frac{m}{E}\right)\left(\Phi_{fi} + \frac{p}{E+m}\,\Phi_{fi}\,\frac{p}{E+m}\right)\cos\frac{\theta}{2} = \Phi_{fi}\cos\frac{\theta}{2},$$

$$\langle E, \boldsymbol{p}_f, - | H_1^{\mathrm{Tr}} | E, \boldsymbol{p}_i, + \rangle = \frac{1}{2}\left(1 + \frac{m}{E}\right)\left(\Phi_{fi} + \frac{-p}{E+m}\,\Phi_{fi}\,\frac{p}{E+m}\right)\left(-\sin\frac{\theta}{2}\right) =$$

$$= -\Phi_{fi}\,\frac{m}{E}\sin\frac{\theta}{2}.$$

By forming the sum of the squares, *i.e.* averaging over the spins along $\boldsymbol{p}_f$, we get the first order Born approximation for the relativistic scattering cross-section (apart from a factor),

$$\sigma(\theta) \sim |\Phi_{fi}|^2\left(\cos^2\frac{\theta}{2} + \frac{m^2}{E^2}\sin^2\frac{\theta}{2}\right) = |\Phi_{fi}|^2\left(1 - \frac{v^2}{c^2}\sin^2\frac{\theta}{2}\right).$$

10. – Charge distributions of the fermion in the new representations.

In their paper FOLDY and WOUTHUYSEN obtain the following transformed Hamiltonian to second order in $1/m$ (ref. [1]), eq. (35), corrected):

$$H^{\mathrm{Tr}} = \beta m + e\Phi + \frac{\beta}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2 - \beta\frac{e}{2m}\boldsymbol{\sigma H} -$$

$$- \frac{e}{8m^2}\boldsymbol{\sigma}(\boldsymbol{E}\times(\boldsymbol{p} - e\boldsymbol{A}) - (\boldsymbol{p} - e\boldsymbol{A})\times\boldsymbol{E}) - \frac{e}{8m^2}\operatorname{div}\boldsymbol{E} + \ldots$$

The last term is a well known correction to the Pauli theory, due to C. G. DARWIN (1928). This term—« which has previously been regarded as of rather mysterious origin, can now easily be understood, since it comes from the fact that the electric charge in the new representation is spread out over a finite region »—of the order of its Compton wave length $\lambda\!\!\!^{-}$ $(= \hbar/mc)$ (*).

In order to see that the potential energy of an extended particle gives rise to a term of this kind, we consider a spherically symmetric charge distribution $\varrho(s)$ about the position $\boldsymbol{r}$. The potential energy of this distribution in the field $\Phi(\boldsymbol{r})$ is given by

$$V(\boldsymbol{r}) = \int \varrho(\boldsymbol{s})\, \Phi(\boldsymbol{r} + \boldsymbol{s})\, \mathrm{d}^3\boldsymbol{s} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^3\boldsymbol{s}\, \varrho(s) (\boldsymbol{s}\nabla)^n \Phi(\boldsymbol{r}) ,$$

which is easily found to be

$$V(\boldsymbol{r}) = \sum_{l=0}^{\infty} \frac{e}{(2l+1)!} \langle s^{2l} \rangle \nabla^{2l} \Phi(\boldsymbol{r}) = e\Phi + \frac{e}{6} \langle s^2 \rangle \nabla^2 \Phi + \dots . \tag{79}$$

$e\langle s^{2l} \rangle$ is the $2l$'th charge moment, given by

$$e\langle s^{2l} \rangle = \int s^{2l}\, \varrho(s)\, \mathrm{d}^3\boldsymbol{s} .$$

Hence, in the F.W. case

$$\langle s^2 \rangle_{\mathrm{FW}} = \tfrac{3}{4} \lambda\!\!\!^{-2} .$$

In order to obtain more detailed knowledge of the charge distribution, one may continue the iteration process, selecting terms of the type $e\nabla^4\Phi$, $e\nabla^6\Phi$, etc. Then, according to (79), higher order charge moments can be determined. To fourth order, for instance, we get

$$V_{\mathrm{FW}}(\boldsymbol{r}) = e\Phi + \tfrac{1}{8} e\lambda\!\!\!^{-2} \nabla^2\Phi + \tfrac{11}{128}\, e\lambda\!\!\!^{-4} \nabla^4 \Phi .$$

These calculations are rapidly becoming very tedious for increasing order of $1/m$, and therefore greatly limit the possibilities of knowing the F.W. charge distribution accurately.

We notice that the energy terms of interest are proportional to the charge e, consequently they are all contained in the first order term of the transformed Hamiltonian, when this is expanded in powers of e. In the F.W. theory, however, the transformed Hamiltonian does not lend itself easily to an expansion of this kind.

(*) F. and W. connect this feature with their new « mean position operator ».

Let us rather examine the charge distributions of the fermion in the representations which are found in this paper. Then we want to pick out from the first order transformed Hamiltonian the part which depends on Φ, and which is free from the spin variables. These terms are, according to the expressions (74) and (73), respectively,

$$W_{\mathrm{I,III}}(\boldsymbol{r}, \boldsymbol{p}) = \frac{e}{2}\left(\Phi + \frac{m}{E_p}\Phi\frac{m}{E_p} + \frac{\boldsymbol{p}}{E_p}\Phi\frac{\boldsymbol{p}}{E_p}\right), \tag{80}$$

$$W_{\mathrm{II}}(\boldsymbol{r}, \boldsymbol{p}) = \frac{e}{2}\sqrt{1+\frac{m}{E_p}}\left(\Phi + \frac{\boldsymbol{p}}{E_p+m}\Phi\frac{\boldsymbol{p}}{E_p+m}\right)\sqrt{1+\frac{m}{E_p}}\,. \tag{81}$$

In order to select the co-ordinate function part of these operators, we apply them to a state with zero momentum, *e.g.* a constant. The result will then be the function which is interpreted as the potential energy $V(\boldsymbol{r})$ of an extended particle:

$$W(\boldsymbol{r}, \boldsymbol{p})\,|\boldsymbol{p}=0\rangle = V(\boldsymbol{r})\,|\boldsymbol{p}=0\rangle\,.$$

In this manner $W_{\mathrm{I,III}}$ and W_{II}, from eq. (80) and (81), give respectively

$$V_{\mathrm{I,III}}(\boldsymbol{r}) = \frac{e}{2}\left(1+\frac{m}{E_p}\right)\Phi(\boldsymbol{r}) \equiv eP_{\mathrm{I,III}}\left(\frac{\boldsymbol{p}}{m}\right)\Phi(\boldsymbol{r})\,, \tag{82}$$

$$V_{\mathrm{II}}(\boldsymbol{r}) = e\sqrt{\frac{1}{2}\left(1+\frac{m}{E_p}\right)}\,\Phi(\boldsymbol{r}) \equiv eP_{\mathrm{II}}\left(\frac{\boldsymbol{p}}{m}\right)\Phi(\boldsymbol{r})\,. \tag{83}$$

The operators $P(\boldsymbol{p}/m)$ are to operate upon $\Phi(\boldsymbol{r})$ only.

Incidentally, from these expressions we may obtain the two sets of charge moments, simply by expanding the operators P in powers of $\boldsymbol{p}/m = -i\lambda\nabla$. We get in the respective representations

$$V_{\mathrm{I,III}}(\boldsymbol{r}) = e\Phi + \tfrac{1}{4}e\lambda^2\nabla^2\Phi + \tfrac{3}{16}e\lambda^4\nabla^4\Phi + \ldots\,,$$

$$V_{\mathrm{II}}(\boldsymbol{r}) = e\Phi + \tfrac{1}{8}e\lambda^2\nabla^2\Phi + \tfrac{11}{128}e\lambda^4\nabla^4\Phi + \ldots\,.$$

As to the transformation $U_{\mathrm{E}} = \sqrt{\beta\lambda}$, it does not give H^{Tr} in powers of e in a simply way, and hence we do not know $W(\boldsymbol{r}, \boldsymbol{p})$. It is, however, possible to find the result of $(H^{\mathrm{Tr}})_1$ applied upon a constant, and from that expression we may pick out the term we are seeking. Putting $\boldsymbol{A} = 0$, we get in fact (*)

$$(H_{\mathrm{E}}^{\mathrm{Tr}})_1\,|\boldsymbol{p}=0\rangle = V_{\mathrm{E}}(\boldsymbol{r})\cdot|\boldsymbol{p}=0\rangle\,,$$

(*) The details will be published elsewhere.

where

$$V_{\mathrm{E}}(\boldsymbol{r}) = e\left\{\sqrt{\frac{2}{1+(m/E_p)}} - \frac{(E_p/m)-1}{(E_p/m)+1}\right\}\Phi(\boldsymbol{r}) \equiv e\,P_{\mathrm{E}}(\boldsymbol{p}/m)\,\Phi(\boldsymbol{r})\,. \tag{84}$$

Expansion in powers of $\boldsymbol{p}/m$ yields

$$V_{\mathrm{E}}(\boldsymbol{r}) = e\Phi + \tfrac{1}{8}e\lambda\!\!\!^{-2}\nabla^2\Phi + \tfrac{7}{128}\lambda\!\!\!^{-4}\nabla^4\Phi + \ldots\,.$$

To find the analytical form of the various charge distributions corresponding to these potential energies, we proceed in the following manner:

$$\begin{aligned} V(\boldsymbol{r}) &= eP(\boldsymbol{p}/m)\,\Phi(\boldsymbol{r}) = eP(\lambda\!\!\!^{-}i\nabla)\int\Phi(\boldsymbol{r}')\,\delta(\boldsymbol{r}'-\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}' = \\ &= e\int\mathrm{d}^3\boldsymbol{r}'\,\Phi(\boldsymbol{r}')\,P(\lambda\!\!\!^{-}i\nabla)(2\pi)^{-3}\int\mathrm{d}^3\boldsymbol{k}\,\exp\,[i(\boldsymbol{r}'-\boldsymbol{r})\boldsymbol{k}] = \\ &= e\int\mathrm{d}^3\boldsymbol{r}'\,\Phi(\boldsymbol{r}')(2\pi)^{-3}\int\mathrm{d}^3\boldsymbol{k}\,P(\lambda\!\!\!^{-}\boldsymbol{k})\,\exp\,[i(\boldsymbol{r}'-\boldsymbol{r})\boldsymbol{k}] = \\ &= \int\mathrm{d}^3\boldsymbol{s}\,\Phi(\boldsymbol{r}+\boldsymbol{s})\,\varrho(\boldsymbol{s})\,, \end{aligned}$$

where $\varrho(\boldsymbol{s})$ is interpreted as the charge density, given by

$$\varrho(\boldsymbol{s}) = \frac{e}{(2\pi\lambda\!\!\!^{-})^3}\int P(\boldsymbol{K})\,\exp\,[i\boldsymbol{S}\boldsymbol{K}]\,\mathrm{d}^3\boldsymbol{K}\,,$$

$$\boldsymbol{S} = \boldsymbol{s}/\lambda\!\!\!^{-}\,.$$

Looking at the expressions for $P(\boldsymbol{p}/m)$ in eq. (82), (83) and (84), we see that $P(\infty) \neq 0$. This means that ϱ is partially a δ-function, *i.e.* a point charge:

$$\varrho = P(\infty)\,e\delta(\boldsymbol{s}) + \frac{e}{(2\pi\lambda\!\!\!^{-})^3}\int[P(\boldsymbol{K}) - P(\infty)]\,\exp\,[i\boldsymbol{S}\boldsymbol{K}]\,\mathrm{d}^3\boldsymbol{K}\,,$$

$$P_{\mathrm{I,III}}(\infty) = \tfrac{1}{2} = 0.5\,,$$

$$P_{\mathrm{II}}(\infty) = \frac{1}{\sqrt{2}} \simeq 0.7\,,$$

$$P_{E}(\infty) = \sqrt{2} - 1 \simeq 0.4\,.$$

The Fourier integrals above can be expressed in terms of well known functions. We obtain the following charge distributions for the fermion:

In picture I and III:

$$\varrho_{\mathrm{I,III}} = \frac{e}{2}\,\delta(\boldsymbol{s}) + \frac{e}{4\pi^2\lambda\bar{}^3}\,\frac{1}{S}\,K_1(S)\,.$$

In picture II:

$$\varrho_{\mathrm{II}} = \frac{e}{\sqrt{2}}\,\delta(\boldsymbol{s}) + \frac{e\sqrt{2}}{8\pi^2\lambda\bar{}^3}\,\frac{1}{S}\,\frac{\mathrm{d}}{\mathrm{d}S}\,\frac{1}{S}\left[\Gamma(\tfrac{5}{4})W_{-\frac{1}{4},\frac{1}{4}}(2S) - \Gamma(\tfrac{3}{4})W_{\frac{1}{4},\frac{1}{4}}(2S)\right].$$

In picture E:

$$\varrho_{\mathrm{E}} = (\sqrt{2}-1)e\,\delta(\boldsymbol{s}) + \frac{e}{\pi^2\lambda\bar{}^3}\,\frac{1}{S}\left[K_1(S) - \int_S^\infty K_0(t)\,\mathrm{d}t\right] +$$

$$+ \frac{e}{2\pi^2\lambda\bar{}^3}\,S^{-\frac{5}{2}}\left[\Gamma(\tfrac{7}{4})W_{-\frac{1}{4},1}(2S) - \Gamma(\tfrac{5}{4})W_{\frac{1}{4},1}(2S)\right].$$

K_ν is the modified Hankel function, and $W_{\varkappa,\mu}$ is Whittaker's function.

In addition to the δ-function singularities, these charge densities have singularities $1/s^2$ in $s = 0$.

The asymptotic forms when $s \to \infty$, are:

$$\varrho_{\mathrm{I,III}} \sim \quad s^{-6/4}\exp\left[-s/\lambda\bar{}\right],$$

$$\varrho_{\mathrm{II}} \sim \quad s^{-7/4}\exp\left[-s/\lambda\bar{}\right],$$

$$\varrho_{\mathrm{E}} \sim -\,s^{-9/4}\exp\left[-s/\lambda\bar{}\right].$$

Appendix

Square root of unitary operator.

Let u be a unitary operator, $uu^* = u^*u = 1$. Then the operator

$$\sqrt{u} = (1+u)(2+u+u^*)^{-\frac{1}{2}},$$

is a solution of the equation $X^2 = u$. This expression is well defined, provided $2+u+u^*$ is a positive definite operator. Let us examine whether this is true. The eigenvalues of u are on the unit circle in the complex plane:

$$u\,|\,\delta\rangle = \exp\left[i\delta\right]|\,\delta\rangle\,,$$

i.e.

$$u^*\,|\,\delta\rangle = \exp\left[-i\delta\right]|\,\delta\rangle\,.$$

Hence:

$$(2 + u + u^*)\,|\,\delta\rangle = 2(1 + \cos\delta)\,|\,\delta\rangle\,.$$

This eigenvalue is positive definite, except when $\cos\delta = -1$. Thus, the operator $(2+u+u^*)^{-\frac{1}{2}}$ (positive root) is well defined when -1 is not an eigenvalue of u. In the following we assume this to be true. Then,

$$\begin{aligned}\sqrt{u}\sqrt{u} &= (1+u)^2\,(2+u+u^*)^{-1} = (1+2u+u^2)(2+u+u^*)^{-1}\,,\\ &= u(u^*+2+u)(2+u+u^*)^{-1} = u\,, \qquad \textit{q.e.d.}\end{aligned}$$

The hermitian conjugate of $\sqrt{u}$ is

$$(\sqrt{u})^* = (1+u^*)(2+u^*+u)^{-\frac{1}{2}} = \sqrt{u^*}\,.$$

As is easily seen $\sqrt{u}$ is unitary:

$$\sqrt{u}\,(\sqrt{u})^* = (\sqrt{u})^*\sqrt{u} = 1\,.$$

Of special interest is the case:

$$u = FG\,,$$

where $F = F^* = F^{-1}$, $G = G^* = G^{-1}$ (for instance, $F = \beta$, $G = \lambda$).

Then,

$$\begin{aligned}\sqrt{FG} &= (1+FG)(2+FG+GF)^{-\frac{1}{2}}\,,\\ (\sqrt{FG})^* &= \sqrt{GF} = (1+GF)(2+GF+FG)^{-\frac{1}{2}}\,.\end{aligned}$$

By means of these expression one can easily verify the following relations:

$$\begin{aligned}F\sqrt{FG} &= \sqrt{GF}\,F\,,\\ \sqrt{GF}\,F\sqrt{FG} &= G\,.\end{aligned}$$

Putting $F = \beta$ and $G = \lambda$, we see that the operator $U = \sqrt{\beta\lambda}$ satisfies the relations

$$\beta U = U^*\beta\,, \qquad U^*\beta U = \lambda\,, \qquad U^*U = UU^* = 1\,,$$

which have been used in the text.

When F and G anticommute, the square root is a very simple expression:

$$\sqrt{FG} = \frac{1}{\sqrt{2}}\,(1+FG)\,.$$

Putting $F = \beta$ and $G = i\beta\gamma_5$, we get

$$\sqrt{\beta\, i\beta\gamma_5} = \sqrt{i\gamma_5} = \frac{1}{\sqrt{2}}\,(1+i\gamma_5)\,.$$

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Statistical Weights in Multiple Particle Production (*).

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CONTENTS. — **1.** Introduction. – **2.** Statement of the problem. – **3.** Derivation of the formula. – **4.** Tables.

1. – Introduction.

In treating the process of multiple production in high energy collisions, statistical weights enter because of the usual assumption of conservation of total isotopic spin. These weights are the product of two factors. One of these factors is the number of independent degenerate states (due to particle exchanges) for a final state described by giving the number of particles of different types and charges. The other is the probability, for a specific state, that a measurement of the square of the total isotopic spin of the system will yield the value corresponding to the initial two particle state.

The purpose of this paper is to derive a simple analytic formula for these weights, valid for arbitrary numbers of particles in the final state, provided that they have isotopic spins 0, $\frac{1}{2}$, or 1. Extensive tables are given for the second factor for those reactions which are experimentally of interest. From these tables, a simple computation gives the statistical weights (equation (5)).

With regard to notation, it should be noted that upper case letters T, T_3, will be used for isotopic spin operators and lower case letters, t, m, for the corresponding quantum numbers. If a symbol is to refer to only a part of the system of particles, a subscript will be appended.

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2. – Statement of the problem.

The reactions being considered are of the type

$$A + B \to \text{many particles} .$$

The cross-section for such a reaction is proportional to the square of the transition matrix element between the initial and final states

$$\sigma \propto |\langle i | \mathscr{I} | f \rangle|^2 . \tag{1}$$

The initial state is either a pure t state (*e.g.* pp collision) or a simple mixture of such states with known probabilities (*e.g.* a $p\pi^0$ system is in either a $t = \frac{1}{2}$ state with a relative probability $\frac{1}{3}$ or in a $t = \frac{3}{2}$ state with a relative probability of $\frac{2}{3}$). Hence, the treatment is restricted to the case of an initial pure t state. If necessary, one can average over such states at the end. The initial state, of course, is always a pure m state.

The final state consists of many non-interacting particles. Consequently, it has a product isotopic spin wave function with a definite value of m, but not of t. (Effects due to antisymmetrization of the fermions will not be considered in the present paper.) This wave function can be expanded in terms of normalized simultaneous eigenfunctions of T and T_3, *i.e.*

$$|f\rangle = |m\rangle = \sum_t c_t |t, m\rangle . \tag{2}$$

If one assumes that t is conserved (charge independence), then the reaction can go only to that term in the expansion corresponding to the t of the initial state. Hence, from (1) and (2)

$$\sigma \propto |c_t|^2 = P_{tm} . \tag{3}$$

A formula for the quantities $P_{t,m}$ has been obtained by CERULUS [1] in terms of Clebsch-Gordon coefficients. For certain cases he has also obtained closed formulae for the $P_{t,m}$ [2]. A simple closed formula will be derived in this

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[1] F. CERULUS: *Suppl. Nuovo Cimento*, **15**, 402 (1960). $P^J_{m_1, \ldots, m_n}$ of this reference is the same as $P_{tm}(\alpha, \beta)$ here. $^*P^J_{m_1, \ldots, m_n}$ of CERULUS is the same as G_{tn}.

[2] F. CERULUS (preprint): This paper gives a general procedure for obtaining closed formulae which yields results that, in appearance, differ appreciably from those given here. He obtains explicit formulae for the case of n pions, n pions and 1 nucleon, and for n pions and 2 nucleons. The author is indebted to Dr. CERULUS for a prepublication copy of this paper.

paper, which is valid for any number of particles in the final state, provided only that they have $t_i = 0$, $\frac{1}{2}$, or 1.

The quantity P_{tm} refers to a single product wave function. Experimentally, however, one is usually interested in a final state described by giving the number of particles of various types and charges, (*e.g.* $p+n+2K^+ +2K^- +\pi^+ +3\pi^0$ from a p+p reaction).

There are many independent product wave functions for such a state, corresponding to particle exchanges. Consider the particles as separated into classes corresponding to the different isotopic spin multiplets, *i.e.* nucleons, antinucleons, pions, etc. Let $n(t_i)$ be the number of particles in class i and $n(t_i, m_i)$ be the number of particles in class i with a specific value m_i for T_{3i}, so that

$$n(t_i) = \sum_{m_i=-t_i}^{t_i} n(t_i, m_i) \,. \tag{4}$$

Then, under the assumption of charge independence, one obtains degenerate independent wave functions under any exchange of two particles in the same class with different values for m_i. Hence, the complete statistical weight for a state described as above is given by

$$G_{tm}[n(t_i, m_i)] = \prod_i \left\{ \frac{[n(t_i)]!}{\prod_{m_i=-t_i}^{t_i} ([n(t_i, m_i)]!)} \right\} P_{tm}[n(t_i, m_i)], \tag{5}$$

where G is a function of all the quantities n. Tables have been given in the literature for P_{tm} and G_{tm} for limited numbers of particles ([1-3]). These tables have been calculated by using known expressions for the Clebsch-Gordon coefficients, and combining the angular momenta two at a time. Extensive tables of P_{tm} are given at the end of this paper. The reason for tabulating P, rather than G, is that (as will be shown) it depends on only t, m and 2 other integers, whereas G depends on the complete set of numbers $n(t_i, m_i)$.

Consider the various reactions leading to the same multiplicity in each class, but differing in the charges of some of the particles (*e.g.* in a pp colli-

([3]) R. H. MILBURN: *Rev. Mod. Phys.*, **27**, 1 (1955); E. FERMI: *Phys. Rev.*, **92**, 452 (1953); **93**, 1434 (1954). There are a few errors in the tables in these papers (see Tables IV and V of MILBURN). The corrected values, with the authors' values given in parentheses, are as follows: final state $(n+++--)$ $G_{\frac{3}{2},-\frac{1}{2}} = 80/35$ (87/35), $G_{\frac{1}{2},-\frac{1}{2}} = 2(123/60)$; final state $(n+00--)$, $G_{\frac{3}{2},-\frac{1}{2}} = 136/35$ (129/35), $G_{\frac{1}{2},-\frac{1}{2}} = 14/5$ (165/60); final state $(pp+0-)$, $G_{1,1} = 12/5$ (154/60); final state $(pn++--)$, $G_{1,1} = 3(175/60)$ and final state $(pn+00)$, $G_{1,1} = 21/10$ (121/60).

sion leading to 2 nucleons and 3 pions, the possibilities $pp\pi^+\pi^0\pi^-$, $pp\pi^0\pi^0\pi^0$, $pn\pi^+\pi^0\pi^0$, $pn\pi^+\pi^+\pi^-$, and $nn\pi^+\pi^+\pi^0$. If one makes the statistical hypothesis that the final states (*i.e.* independent wave functions) for these reactions have equal weight, then the quantities G_{tm} are equal to the branching ratios between the above processes. The sum of the statistical weights G_{tm} over the above set of reactions equals the total number of independent isotopic spin wave functions which can be constructed leading to the specified values of t and m. Alternatively, this sum is the total number of multiplets with the given value of t for a system with the specified number of particles in each class. Tables of these quantities have been published in the literature ([4,5]). These can be obtained from explicit formulae ([4]) or from simple recurrence relations obtained from the rule for combination of angular momenta ([5]).

3. – Derivation of the formula.

A closed expression for P_{tm} will now be derived, making use of angular momentum projection operators. Since the result is invariant under a reflection in the 1, 2 plane in isotopic sipn space, one can assume $m \geqslant 0$. The values for negative m are given by

$$P_{t,-|n|}[n(t_i, -m_i)] = P_{t,|n|}[n(t_i, m_i)] . \tag{6}$$

An alternative expansion of the final state wave function in terms of t states is

$$|f\rangle = |m\rangle = \sum_t O_{tm}|m\rangle , \tag{7}$$

where O_{tm} is the projection operator giving a state described by t and m from state described by m. Comparing with (2), one obtains

$$c_t|t, m\rangle = O_{tm}|m\rangle . \tag{8}$$

Since the projection operators satisfy the relation

$$O_{tm}^2 = O_{tm} , \tag{9}$$

and since the states are normalized, it follows that

$$P_{tm} = |c_t|^2 = \langle m|O_{tm}|m\rangle . \tag{10}$$

([4]) Y. Yeivin and A. de-Shalit: *Nuovo Cimento*, **1**, 1146 (1955).
([5]) V. S. Barašenkov and B. M. Barbašev: *Suppl. Nuovo Cimento*, **7**, 19 (1957).

The explicit formula for the projection operators is ([6,7])

$$O_{tm} = \prod_{\substack{\nu=m \\ \nu \neq t}}^{\bar{t}} \frac{T^2 - \nu(\nu+1)}{t(t+1) - \nu(\nu+1)}, \tag{11}$$

where $\bar{t}$ is the maximum possible value of t, which here is just the sum of the t_i values for the individual particles in the final state. It has been shown by LÖWDIN ([8]) that formula (11) can be rewritten as a sum involving the usual raising and lowering operators

$$O_{tm} = \frac{(2t+1)(t+m)!}{(t-m)!} \sum_{\nu=0}^{\bar{t}-t} \frac{(-1)^\nu {}_t T_-^{t-m+\nu} T_+^{t-m+\nu}}{\nu!(2t+\nu+1)!}. \tag{12}$$

This is valid for $m \geqslant 0$. For $m < 0$, one has the above expression with m changed into $-m$ and with the lowering operators to the right. Substituting (12) into (10) yields

$$P_{tm} = \frac{(2t+1)(t+m)!}{(t-m)!} \sum_{\nu=0}^{\bar{t}-t} \frac{(-1)^\nu \langle m | T_-^{t-m+\nu} T_+^{t-m+\nu} | m \rangle}{\nu!(2t+\nu+1)!}. \tag{13}$$

The problem is now reduced to evaluating the diagonal matrix elements in equation (13), between product wave functions. The (t_i, m_i) values for the individual particles, will be assumed to be restricted to the 6 sets (1, 1) (1,0), (1, −1), $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$ and (0, 0).

In equation (13) one can think of the operators as being expanded in terms of those for single particles, *i.e.*

$$T_\pm = \sum_i T_{i\pm}. \tag{14}$$

Since raising a state of maximum projection annihilates it, and since the raising operators appear to the right in (13), it follows that P does not depend on any particles with $m = t$. Thus P depends on only the number of particles of the types (1, 0), (1, −1), $(\frac{1}{2}, -\frac{1}{2})$. Further, the wave function in isotopic spin space for a (1, −1) particle behaves, under rotations, exactly like the product wave function for two particles of type $(\frac{1}{2}, -\frac{1}{2})$. Since one is considering product wave functions here, each (1, −1) particle is equivalent

([6]) P. A. M. DIRAC: *The Principles of Quantum Mechanics*, 3rd ed. (Oxford), p. 32.

([7]) P. O. LÖWDIN: *Phys. Rev.*, **97**, 1509 (1955); *Adv. Physics*, **5**, 1 (1956).

([8]) P. O. LÖWDIN: *Angular Momenta Wave Functions Constructed by Projection Operators; Technical note from the Quantum Chemistry Group of Uppsala University*, May 10, 1958.

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to two $(\frac{1}{2}, -\frac{1}{2})$ particles. Thus, if one defines

$$\begin{cases} \alpha = n(1, 0)\,, \\ \beta = n(\frac{1}{2}, -\frac{1}{2}) + 2n(1, -1)\,, \end{cases} \tag{15}$$

(here $n(1, 0)$ means the total number of particles with $t_i = 1$, $m_i = 0$) then P for the actual particles in the final state is the same as for a set of α particles of type $(1, 0)$ and β-particles of type $(1, -1)$. Henceforth, this equivalent set of particles will be used and P_{tm} will be written as $P_{tm}(\alpha, \beta)$ to show that it depends on only α and β.

Next, one expands

$$T_\pm = T_{\alpha\pm} + T_{\beta\pm}\,, \tag{16}$$

where the operators on the right are the sums of the single particle operators for the α and β particle classes, respectively. Using the binomial theorem

$$T_\pm^{t-m+\nu} = \sum_i \binom{t-m+\nu}{i} T_{\alpha\pm}^i \, T_{\beta\pm}^{t-m+\nu-i}\,, \tag{17}$$

and recalling that one has product wave functions and that each particle must be raised and lowered the same number of times, one obtains for the matrix elements in (13) the result

$$\begin{aligned} \langle m | T_-^{t-m+\nu} T_+^{t-m+\nu} | m \rangle = \\ = \sum_i \binom{t-m+\nu}{i}^2 \langle \alpha | T_{\alpha-}^i T_{\alpha+}^i | \alpha \rangle \langle \beta | T_{\beta-}^{t-m+\nu} T_{\beta+}^{t-m+\nu} | \beta \rangle\,, \end{aligned} \tag{18}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are the product wave functions for the two types of particles.

Again, think of the operators in (18) as being expanded in terms of those for individual particles. Since, for both classes of particles, $m_i = t_i - 1$, one is concerned only with single raisings and lowerings (or no raisings, which just gives a factor unity). Since

$$\begin{cases} t_{i-} t_{i+} | 1, 0 \rangle = 2 | 1, 0 \rangle\,, \\ t_{1-} t_{i+} | \frac{1}{2}, -\frac{1}{2} \rangle = | \frac{1}{2}, -\frac{1}{2} \rangle\,, \end{cases} \tag{19}$$

it follows that to evaluate the matrix elements in (18) one has to merely count the number of possible ways of raising and lowering once, and multiply by

the appropriate power of 2 or 1. Consider the α matrix elements, assuming the operators expanded. One is raising and lowering i particles of type (1, 0), giving a factor 2^i. There are $\binom{\alpha}{i}$ ways of selecting i out of α particles. For each combination one has $i!$ possible permutations of the factors in $T^i_{\nu+}$ and $i!$ possible permutations of the factors in $T^i_{\alpha-}$. Hence

$$\langle\alpha|\, T^i_{\alpha-}\, T^i_{\alpha+}\,|\alpha\rangle = \frac{i!\,\alpha!}{(\alpha-i)!}\,2^i\,. \tag{20}$$

Similarly

$$\langle\beta|\, T^{t-m+\nu-i}_{\beta-}\, T^{t-m+\nu-i}_{\beta+}\,|\beta\rangle = \frac{(t-m+\nu-i)!\,\beta!}{(\beta-t+m-\nu+i)!}\,. \tag{21}$$

Combining (13), (18), (20) and (21), one obtains finally

$$P_{tm}(\alpha,\beta) = \frac{(2t+1)(t+m)!}{(t-m)!}\sum_{\nu}\sum_{i}\frac{(-1)^\nu[(t-m+\nu)!]^2}{\nu!(2t+\nu+1)!}\binom{\alpha}{i}\binom{\beta}{t-m+\nu-i}2^i\,. \tag{22}$$

The limits on the summation are such that the arguments of all factorials are greater than or equal to zero. Equation (22) can be written in a more symmetrical form by the substitution $\lambda = t-m+\nu-i$ leading to

$$P_{tm}(\alpha,\beta) = \frac{(-1)^{t-m}(2t+1)(t+m)!}{(t-m)!}\cdot \sum_{\substack{i=0}}^{\alpha}\sum_{\substack{\lambda=0\\(i+\lambda\geq t-m)}}^{\beta}\frac{(-1)^{i+\lambda}[(i+\lambda)!]^2}{(i+\lambda-t+m)!\,(t+m+1+i+\lambda)!}\binom{\alpha}{i}\binom{\beta}{\lambda}2^i\,. \tag{23}$$

Here the limits of summation have been written explicitly. The lower limits can be written as zero, if desired, since the additional terms vanish because of the poles in $(i+\lambda-t+m)!$.

For large numbers of particles, (*i.e.* large α and β) equations (22) and (23) are inconvenient to use because of the large number of terms involved. A more useful formula can be derived as follows

Recall the well known property of the hypergeometric functions of argument unity,

$$F(a,b,c,1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,, \tag{24}$$

provided

$$\mathrm{Re}\,(c-a-b) > 0 \qquad\qquad c \neq 0, -1, -2 \ldots .$$

Considering (24) as a complex function of c, one can readily show that for

p, n, k integers satisfying $k \geqslant n \geqslant p \geqslant 0$

$$F(-p, -n, -k, 1) = \frac{(k-n) \ldots (k-p-n+1)}{k(k-1) \ldots (k-p+1)} .$$

If $k < p+n$, the right hand side of this expression vanishes. Using the polynomial expansion for the hypergeometric function, one readily obtains

$$\sum_{s=0}^{p} \frac{(-1)^s (k-p+s)!}{s!\,(p-s)!\,(n-p+s)!} = \frac{(-1)^p (k-p)!\,(k-n)!}{(k-p-n)!\,p!\,n!} . \tag{25}$$

Setting $p = n = t-m$, $k = i+\lambda+t-m$, equation (25) becomes

$$\frac{(-1)^{t-m}[(i+\lambda)!]^2}{(i+\lambda-t+m)![(t-m)!]^2} = \sum_{s=0}^{t-m} \frac{(-1)^s (i+\lambda+s)!}{(s!)^2 (t-m-s)!} . \tag{26}$$

This is valid for $i \geqslant 0$, $j \geqslant 0$. For $0 \leqslant i+\lambda < t-m$, both sides of equation (26) vanish. Hence, this can be substituted into (23) and the lower limits on the sums over i and λ taken as zero. One obtains

$$P_{tm}(\alpha, \beta) = (2t+1)(t+m)!\,(t-m)!\cdot$$
$$\cdot \sum_{s=0}^{t-m} \sum_{i=0}^{\alpha} \sum_{\lambda=0}^{\beta} \frac{(-1)^{i+\lambda+s} (i+\lambda+s)!}{(s!)^2 (t-m-s)!\,(t+m+1+i+\lambda)!} \binom{\alpha}{i} \binom{\beta}{\lambda} 2^i .$$

In this, the sum over λ can be performed using (24) yielding

$$P_{tm}(\alpha, \beta) = (2t+1)(t+m)!\,(t-m)!\cdot$$
$$\cdot \sum_{s=0}^{t-m} \sum_{i=0}^{\alpha} \frac{(-1)^{i+s} (t+m+\beta-s)!\,(i+s)!}{(s!)^2 (t-m-s)!\,(t+m-s)!\,(t+m+\beta+1+i)!} \binom{\alpha}{i} 2^i .$$

Here, the sum over i reduces to a hypergeometric function of argument 2, yielding finally

$$P_{tm}(\alpha, \beta) = \frac{(2t+1)(t+m)!\,(t-m)!}{(t+m+\beta+1)!} \sum_{s=0}^{t-m} \frac{(-1)^s (t+m+\beta-s)!}{(s!)\,(t-m-s)!\,(t+m-s)!} \cdot \tag{27}$$
$$\cdot F(-\alpha, 1+s, t+m+\beta+2, 2) .$$

Equation (22), (23) or (27) gives an analytic expression for the probability $P_{tm}(\alpha, \beta)$. The statistical weight factor is then given by equation (5)

$$G_{tm}[n(t_i, m_i)] = \prod_i \left\{ \frac{[n(t_i)]!}{\prod_{m_i=-t_i}^{t_i} ([n(t_i, m_i)]!)} \right\} P_{tm}(\alpha\beta) , \tag{5}$$

which has been rewritten to emphasize that P_{tm} depends on only α and β, which are defined by equation (15).

The above results are valid for $m \geqslant 0$. For $m < 0$, if one defines

$$\beta' = n(\tfrac{1}{2}, \tfrac{1}{2}) + 2n(1, 1) \tag{28}$$

and α as before, the probabilities are given by

$$P_{t,-|m|}(\alpha, \beta') = P_{t,|m|}(\alpha, \beta' = \beta)\,. \tag{29}$$

Since the hypergeometric functions can be obtained easily by recurrence relations, (25) is by far the simplest form for computing the probabilities. Furthermore, in those reactions observed up to the present, one particle is always a nucleon, so that the maximum value of t is $\frac{3}{2}$. Thus, one is interested only in the two cases $m = t$ and $m = t - 1$. For these cases, (27) reduces to

$$P_{t,t}(\alpha, \beta) = \frac{2t+1}{2t+1+\beta} F(-\alpha, 1, 2t+2+\beta, 2)\,, \tag{30}$$

$$P_{t,t-1}(\alpha, \beta) = \tag{31}$$

$$= \frac{2t+1}{2t+\beta}\left[F(-\alpha, 1, 2t+1+\beta, 2) - \frac{2t-1}{2t-1+\beta} F(-\alpha, 2, 2t+1+\beta, 2)\right].$$

Perhaps a few words are in order regarding possible particles with higher isotopic spins. Although the results given here are restricted to $t_i \leqslant 1$, they include higher spin particles provided they have maximum or minimum projection (*i.e.* $m_i = t_i$ or $m_i = -t_i$). This follows because in these cases the wave functions behave as product wave functions of spin $\frac{1}{2}$ particles. For example, consider a $(\frac{3}{2}, \frac{3}{2})$ particle. For m positive, it has no effect on P_{tm}; for m negative one simply increases β' by 3.

4. – Tables.

Tables have been computed for $P_{tm}(\alpha, \beta)$ for the six sets of (t, m) values $(1, 1)$, $(1, 0)$, $(0, 0)$, $(\frac{3}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$. These were computed as decimal fractions on the Bendix G-15 computer at Fordham University, using equations (30) and (31). Table I gives $P_{tm}(\alpha, \beta)$ for these six states for the range of variables $0 \leqslant \alpha \leqslant 9$, $0 \leqslant \beta \leqslant 18$. Some tables of P_{tm} as rational fractions are given in reference [1].

A simple example will illustrate the use of these tables. Consider the

TABLE I. – *Values of* $P_{tm}(\alpha, \beta)$. — $P_{tm}(\alpha, \beta)$ *for state* (t, m).

α	β	(1, 1)	(1, 0)	(0, 0)	$(\frac{3}{2}, \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
0	0	1.0	0	1.0	1.0	0	1.0
	1	.75	.5	.5	.8	.333 33	.666 67
	2	.6	.5	.333 33	.666 67	.4	.5
	3	.5	.45	.25	.571 43	.4	.4
	4	.428 57	.4	.2	.5	.380 95	.333 33
	5	.375	.357 14	.166 67	.444 44	.357 14	.285 71
	6	.33 333	.321 43	.142 86	.4	.333 33	.25
	7	.3	.291 67	.125	.363 64	.311 11	.222 22
	8	.272 73	.266 67	.111 11	.333 33	.290 91	.2
	9	.25	.245 45	.1	.307 69	.272 73	.181 82
	10	.230 77	.227 27	.090 91	.285 71	.256 41	.166 67
	11	.214 29	.211 54	.083 33	.266 67	.241 76	.153 85
	12	.2	.197 80	.076 92	.25	.228 57	.142 86
	13	.187 5	.185 71	.071 43	.235 29	.216 67	.133 33
	14	.176 47	.175	.066 67	.222 22	.205 88	.125
	15	.166 67	.165 44	.062 5	.210 53	.196 08	.117 65
	16	.157 89	.156 86	.058 82	.2	.187 13	.111 11
	17	.15	.149 12	.055 56	.190 48	.178 95	.105 26
	18	.142 86	.142 11	.052 63	.181 82	.171 43	.1
1	0	.5	1.0	0	.6	.666 67	.333 33
	1	.45	.5	.166 67	.533 33	.466 67	.333 33
	2	.4	.4	.166 67	.476 19	.4	.3
	3	.357 14	.35	.15	.428 57	.361 90	.266 67
	4	.321 43	.314 29	.133 33	.388 89	.333 33	.238 10
	5	.291 67	.285 71	.119 05	.355 56	.309 52	.214 29
	6	.266 67	.261 90	.107 14	.327 27	.288 89	.194 44
	7	.245 45	.241 67	.097 22	.303 03	.270 71	.177 78
	8	.227 27	.224 24	.088 89	.282 05	.254 55	.163 64
	9	.211 54	.209 09	.081 82	.263 74	.240 09	.151 52
	10	.197 80	.195 80	.075 76	.247 62	.227 11	.141 03
	11	.185 71	.184 07	.070 51	.233 33	.215 38	.131 87
	12	.175	.173 63	.065 93	.220 59	.204 76	.123 81
	13	.165 44	.164 29	.061 90	.209 15	.195 10	.116 67
	14	.156 86	.155 88	.058 33	.198 83	.186 27	.110 29
	15	.149 12	.148 28	.055 15	.189 47	.178 19	.104 58
	16	.142 11	.141 38	.052 29	.180 95	.170 76	.099 42
	17	.135 71	.135 09	.049 71	.173 16	.163 91	.094 74
	18	.129 87	.129 32	.047 37	.166 01	.157 58	.090 48

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TABLE I (*continued*).

α	β	(1, 1)	(1, 0)	(0, 0)	$(\frac{3}{2}, \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
2	0	.4	0	.333 33	.466 67	.266 67	.333 33
	1	.35	.3	.166 67	.419 05	.333 33	.266 67
	2	.314 29	.3	.133 33	.380 95	.323 81	.233 33
	3	.285 71	.278 57	.116 67	.349 21	.304 76	.209 52
	4	.261 90	.257 14	.104 76	.322 22	.285 71	.190 48
	5	.241 67	.238 10	.095 24	.298 99	.268 25	.174 60
	6	.224 24	.221 43	.087 30	.278 79	.252 53	.161 11
	7	.209 09	.206 82	.080 56	.261 07	.238 38	.149 49
	8	.195 80	.193 94	.074 75	.245 42	.225 64	.139 39
	9	.184 07	.182 52	.069 70	.231 50	.214 12	.130 54
	10	.173 63	.172 33	.065 27	.219 05	.203 66	.122 71
	11	.164 29	.163 19	.061 36	.207 84	.194 14	.115 75
	12	.155 88	.154 94	.057 88	.197 71	.185 43	.109 52
	13	.148 28	.147 48	.054 76	.188 51	.177 45	.103 92
	14	.141 38	.140 69	.051 96	.180 12	.170 11	.098 86
	15	.135 09	.134 48	.049 43	.172 43	.163 33	.094 26
	16	.129 32	.128 79	.047 13	.165 37	.157 06	.090 06
	17	.124 03	.123 56	.045 03	.158 86	.151 24	.086 22
	18	.119 14	.11873	.043 11	.15283	.145 83	.082 68
3	0	.3	.6	0	.371 43	.4	.2
	1	.278 57	.3	.1	.342 86	.314 29	.2
	2	.257 14	.257 14	.1	.317 46	.285 71	.185 71
	3	.238 10	.235 71	.092 86	.295 24	.266 67	.171 43
	4	.221 43	.219 05	.085 71	.275 76	.250 79	.158 73
	5	.206 82	.204 76	.079 36	.258 59	.236 80	.147 62
	6	.193 94	.192 21	.073 81	.243 36	.224 24	.137 88
	7	.182 52	.181 06	.068 94	.229 77	.212 90	.129 29
	8	.172 33	.171 10	.064 65	.217 58	.202 60	.121 68
	9	.163 19	.162 14	.060 84	.206 59	.193 21	.114 89
	10	.154 94	.154 05	.057 44	.196 64	.184 62	.108 79
	11	.147 48	.146 70	.054 40	.187 58	.176 73	.103 30
	12	.140 69	.140 01	.051 65	.179 31	.169 47	.098 32
	13	.134 48	.133 89	.049 16	.171 72	.162 76	.093 79
	14	.128 79	.128 28	.046 90	.164 75	.156 55	.089 65
	15	.123 56	.123 10	.044 83	.158 30	.150 79	.085 86
	16	.118 73	.118 33	.042 93	.152 34	.145 42	.082 37
	17	.114 26	.113 90	.041 19	.146 81	.140 42	.079 15
	18	.110 11	.109 79	.039 58	.141 66	.135 74	.076 17

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TABLE I (*continued*).

α	β	(1, 1)	(1, 0)	(0, 0)	$(\frac{3}{2}, \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
4	0	.257 14	0	.2	.314 29	.228 57	.2
	1	.235 71	.214 29	.1	.292 06	.257 14	.171 43
	2	.219 05	.214 29	.085 71	.273 02	.247 62	.157 14
	3	.204 76	.203 38	.078 57	.256 28	.234 92	.146 03
	4	.192 21	.190 48	.073 02	.241 41	.222 80	.136 51
	5	.181 06	.179 65	.068 25	.228 13	.211 69	.128 14
	6	.171 10	.169 91	.064 07	.216 18	.201 55	.120 71
	7	.162 14	.161 13	.060 35	.205 39	.192 30	.114 06
	8	.154 05	.153 18	.057 03	.195 60	.183 82	.108 09
	9	.146 70	.145 95	.054 05	.186 68	.176 02	.102 70
	10	.140 01	.139 36	.051 35	.178 52	.168 84	.097 80
	11	.133 89	.133 32	.048 90	.171 04	.162 21	.093 34
	12	.128 28	.127 77	.046 67	.164 14	.156 06	.089 26
	13	.123 10	.122 66	.044 63	.157 77	.150 34	.085 52
	14	.118 33	.117 93	.042 76	.151 86	.145 02	.082 07
	15	.113 90	.113 55	.041 03	.146 38	.140 06	.078 88
	16	.109 79	.109 47	.039 44	.141 28	.135 42	.075 93
	17	.105 96	.105 67	.037 97	.136 51	.131 06	.073 19
	18	.102 38	.102 13	.036 60	.132 05	.126 98	.070 64
5	0	.214 29	.428 57	0	.269 84	.285 71	.142 86
	1	.202 38	.214 29	.071 43	.253 97	.238 10	.142 86
	2	.190 48	.190 48	.071 43	.239 54	.222 22	.134 92
	3	.179 65	.178 57	.067 46	.226 55	.210 68	.126 98
	4	.169 91	.168 83	.063 49	.214 84	.200 58	.119 77
	5	.161 13	.160 17	.059 88	.204 24	.191 42	.113 28
	6	.153 18	.152 35	.056 64	.194 61	.183 04	.107 42
	7	.145 95	.145 23	.053 71	.185 81	.175 34	.102 12
	8	.139 36	.138 73	.051 06	.177 76	.168 23	.097 30
	9	.133 32	.132 77	.048 65	.170 37	.161 66	.092 91
	10	.127 77	.127 28	.046 45	.163 55	.155 57	.088 88
	11	.122 66	.122 23	.044 44	.157 24	.149 91	.085 18
	12	.117 93	.117 54	.042 59	.151 40	.144 63	.081 77
	13	.113 55	.113 20	.040 89	.145 96	.139 70	.078 62
	14	.109 47	.109 16	.039 31	.140 90	.135 09	.075 70
	15	.105 67	.105 40	.037 85	.136 17	.130 77	.072 98
	16	.102 13	.101 88	.036 49	.131 74	.126 71	.070 45
	17	.098 81	.098 58	.035 22	.127 59	.122 89	.068 09
	18	.095 70	.095 49	.034 04	.123 69	.119 29	.065 87

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TABLE I (*continued*).

α	β	(1, 1)	(1, 0)	(0, 0)	$(\frac{3}{2}, \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
6	0	.190 48	0	.142 86	.238 10	.190 48	.142 86
	1	.178 57	.166 67	.071 43	.225 11	.206 35	.126 98
	2	.168 83	.166 67	.063 49	.213 56	.199 13	.119 05
	3	.160 17	.159 09	.059 52	.203 13	.190 48	.112 55
	4	.152 35	.151 52	.056 28	.193 64	.182 26	.106 78
	5	.145 23	.144 52	.053 39	.184 97	.174 66	.101 57
	6	.138 73	.138 11	.050 78	.177 02	.167 63	.096 82
	7	.132 77	.132 23	.048 41	.169 71	.161 13	.092 49
	8	.127 28	.126 81	.046 24	.162 97	.155 09	.088 51
	9	.122 23	.121 80	.044 26	.156 73	.149 48	.084 86
	10	.117 54	.117 17	.042 43	.150 94	.144 24	.081 48
	11	.113 20	.112 86	.040 74	.145 55	.139 36	.078 36
	12	.109 16	.108 86	.039 18	.140 53	.134 78	.075 47
	13	.105 40	.105 12	.037 73	.135 84	.130 48	.072 78
	14	.101 88	.101 63	.036 39	.131 44	.126 45	.070 26
	15	.098 58	.098 36	.035 13	.127 32	.122 65	.067 92
	16	.095 49	.095 29	.033 96	.123 44	.119 07	.065 72
	17	.092 58	.092 40	.032 86	.119 79	.115 69	.063 66
	18	.089 85	.089 68	.031 83	.116 35	.112 49	.061 72
7	0	.166 67	.333 33	0	.212 12	.222 22	.111 11
	1	.159 09	.166 67	.055 56	.202 02	.191 92	.111 11
	2	.151 52	.151 52	.055 56	.192 70	.181 82	.106 06
	3	.144 52	.143 94	.053 03	.184 15	.174 05	.101 01
	4	.138 11	.137 53	.050 51	.176 30	.167 06	.096 35
	5	.132 23	.131 70	.048 17	.169 08	.160 61	.092 07
	6	.126 81	.126 34	.046 04	.162 40	.154 62	.088 15
	7	.121 80	.121 39	.044 08	.156 22	.149 06	.084 54
	8	.117 17	.116 80	.042 27	.150 49	.143 86	.081 20
	9	.112 86	.112 53	.040 60	.145 15	.139 01	.078 11
	10	.108 86	.108 56	.039 06	.140 16	.134 47	.075 24
	11	.105 12	.104 85	.037 62	.135 51	.130 20	.072 57
	12	.101 63	.101 39	.036 29	.131 14	.126 19	.070 08
	13	.098 36	.098 14	.035 04	.127 05	.122 42	.067 75
	14	.095 29	.095 09	.033 88	.123 20	.118 86	.065 57
	15	.092 40	.092 21	.032 79	.119 57	.115 49	.063 52
	16	.089 68	.089 51	.031 76	.116 14	.112 31	.061 60
	17	.087 11	.086 95	.030 80	.112 91	.109 30	.059 78
	18	.084 68	.084 54	.029 89	.109 85	.106 44	.058 07

TABLE I (*continued*).

α	β	(1, 1)	(1, 0)	(0, 0)	$(\frac{3}{2}, \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
8	0	.151 52	0	.111 11	.191 92	.161 62	.111 11
	1	.143 94	.136 36	.055 56	.183 37	.171 72	.101 01
	2	.137 53	.136 36	.050 51	.175 60	.166 28	.095 96
	3	.131 70	.131 12	.047 98	.168 45	.160 06	.091 69
	4	.126 34	.125 87	.045 84	.161 85	.154 16	.087 80
	5	.121 39	.120 98	.043 90	.155 73	.148 64	.084 23
	6	.116 80	.116 43	.042 11	.150 04	.143 49	.080 92
	7	.112 53	.112 21	.040 46	.144 75	.138 67	.077 86
	8	.108 56	.108 27	.038 93	.139 81	.134 16	.075 02
	9	.104 85	.104 59	.037 51	.135 18	.129 92	.072 37
	10	.101 39	.101 15	.036 19	.130 85	.125 94	.069 90
	11	.098 14	.097 92	.034 95	.126 78	.122 18	.067 59
	12	.095 09	.094 89	.033 80	.122 95	.118 64	.065 42
	13	.092 21	.092 03	.032 71	.119 34	.115 30	.063 39
	14	.089 51	.089 34	.031 70	.115 94	.112 13	.061 48
	15	.086 95	.086 80	.030 74	.112 72	.109 13	.059 67
	16	.084 54	.084 40	.029 84	.109 67	.106 28	.057 97
	17	.082 25	.082 13	.028 98	.106 78	.103 58	.056 36
	18	.080 09	.079 97	.028 18	.104 04	.101 01	.054 84
	0	.136 36	.272 73	0	.174 83	.181 82	.090 91
	1	.131 12	.136 36	.045 45	.167 83	.160 84	.090 91
	2	.125 87	.125 87	.045 45	.161 31	.153 85	.087 41
	3	.120 98	.120 63	.043 71	.155 24	.148 25	.083 92
	4	.116 43	.116 08	.041 96	.149 61	.143 12	.080 65
	5	.112 21	.111 89	.040 33	.144 36	.138 34	.077 62
	6	.108 27	.107 98	.038 81	.139 45	.133 85	.074 80
	7	.104 59	.104 33	.037 40	.134 86	.129 64	.072 18
	8	.101 15	.100 91	.036 09	.130 56	.125 68	.069 73
	9	.097 92	.097 70	.034 86	.126 52	.121 95	.067 43
	10	.094 89	.094 69	.033 72	.122 71	.118 43	.065 28
	11	.092 03	.091 85	.032 64	.119 12	.115 10	.063 26
	12	.089 34	.089 18	.031 63	.115 74	.111 95	.061 36
	13	.086 80	.086 65	.030 68	.112 53	.108 96	.059 56
	14	.084 40	.084 26	.029 78	.109 50	.106 13	.057 87
	15	.082 13	.082 00	.028 93	.106 63	.103 44	.056 27
	16	.079 97	.079 85	.028 13	.103 89	.100 87	.054 75
	17	.077 92	.077 81	.027 38	.101 30	.098 43	.053 31
	18	.075 97	.075 87	.026 66	.098 83	.096 11	.051 95

reaction

$$n + p \to n + 3p + 2\bar{p} + 3\pi^+ + 2\pi^0 + 3\pi^- .$$

Here $\alpha = 2$, $\beta = 3+2(3) = 9$. From the tables $P_{1,0}(2, 9) = .182\,52$

$$P_{0,0}(2, 9) = .069\,70 .$$

Hence $P_{np}(2, 9) = \frac{1}{2}(.182\,52) + \frac{1}{2}(.069\,70) = .126\,11$ and

$$G = \frac{4!}{3!\,1!}\,\frac{2!}{2!}\,\frac{8!}{3!\,2!\,3!}\,(.126\,11) = 2\,240\,(.126\,11) = 282.49 .$$

* * *

The author would like to express his appreciation to Rev. Professor J. McConnel for many stimulating discussions and suggestions and also for checking the derivations in this paper.

SUPPLEMENTO AL VOLUME XVIII, SERIE X
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4° Trimestre

Dislocation Relaxation in Silver, Gold, Palladium and Platinum.

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CONTENTS. — **1. Introduction.** – **2. Results** – **3. Discussion.** 1. Activation energy. 2. Relaxation spectra. 3. Frequency relaxation. 4. Influence of anneal and cold work. 5. Subsidiary peaks. – **4. Conclusions.**

1. – Introduction.

The existence of a relaxation effect due to the dislocations in silver, gold, palladium and platinum has already been reported in a preliminary communication to the Third International Congress of Acoustic (Stuttgart, 1959) (1). In the present paper a larger amount of experimental evidence concerning the same metals is analysed in order to obtain a more complete knowledge of the effect, as it was previously done with copper (2).

In addition to the fundamental parameters (activation energy and characteristic time) whose values are discussed in the light of the present theories of dislocation motion, many other aspects of the effect have been considered, such as the width of the relaxation spectrum, the relation between the total

(1) P. G. BORDONI, M. NUOVO and L. VERDINI: *Relaxation of dislocations in face-centered cubic metals*, in *Proc. of the Third International Congress of Acoustics* (Stuttgart, September 1959), in press.

(2) P. G. BORDONI, M. NUOVO and L. VERDINI: *Nuovo Cimento*, **14**, 273 (1959).

number of dislocations and the characteristic time, the influence of anneal and cold work upon the total number of active dislocations and upon the width of the relaxation spectrum. Much attention has also been given to the subsidiary or Niblett and Wilks peaks, which seem to be a general if not very evident feature of the relaxation effect.

It may be hoped that the considerable amount of experimental evidence which is now available shall provide a good basis for a more detailed model of dislocation motion, and for a better theoretical treatment of the relaxation effect.

2. – Results.

The measurements have been made on circular polycrystalline plates of chemically pure silver, gold, palladium and platinum whose main characteristics are listed in Table I together with their thermal and mechanical history ([3]). The experimental technique is the same used in the previous investigation on copper ([4]).

The energy dissipation and the resonant frequency have been measured as a function of the temperature T in the range $(60 \div 300)$ °K for the two lowest flexural symmetrical modes of the plates, having respectively one and two nodal circles and no nodal diameters. On the *same* specimens the dissipation and the resonant frequency have also been measured for one or two longitudinal modes. These measurements have proved more difficult than the corresponding ones on copper, owing to the coarser grain structure of the present specimens. It has been found that the average grain size may be of the order of some millimeters when the material is machined from the cast. When the wavelength is of the same order of magnitude, the amplitude *vs.* frequency curve is no longer bell-shaped but exhibits side peaks and the measurement of dissipation is no longer possible. The above difficulty has been overcome by obtaining a finer grain structure whenever possible by cold-rolling and intermediate annealing (silver no. 3, Table I) and in every case by measuring the dissipation for longitudinal waves with wavelengths larger than 4.5 mm.

([3]) The authors wish to thank the « Ditta E. Longobardi - Metalli Preziosi » for its co-operation in the preparation of the samples and in their analysis. They also express their gratitude to Prof. D. GENTILONI and to Mr. MACCARRONE of the Istituto di Scienza delle Costruzioni of the University of Rome for their kind help in the mechanical treatment of the specimens.

([4]) See the paper quoted in footnote ([2]). Additional information on the experimental apparatus may be found in the following papers: P. G. BORDONI: *Nuovo Cimento*, **4**, 177 (1947); *Ric. Scient.*, **18**, 103 (1948); P. G. BORDONI and M. NUOVO: *Acustica*, **7**, 1 (1957); *Ric. Scient.*, **27**, 695 (1957); *Acustica*, **8**, 351 (1958).

In platinum an intermediate vibration mode has also been employed for dissipation measurements.

TABLE I. – *Characteristics of the specimens.*

Metal	Purity	Specimen	Thermal and mechanical treatments	Diameter (mm)	Thickness (mm)
Silver	0.9980	1	(*a*) Cross rolled after casting from 12 mm to 10 mm; machined.	36.0	10.00
		2	(*a*) Cross rolled after casting from 12 mm to 10 mm; machined.	29.0	5.00
		3	(*b*) Cold rolled from 10 mm to 7.5 mm; annealed 1 h at 900 °K; cold rolled from 7.5 to 5.0 mm; machined. (*c*) Annealed 3 h at 460 °K. (*d*) Annealed 4 h at 493 °K. (*e*) Permanently strained 2.7% at room temperature.	36.0 (*)	4.55
Gold	0.9988	—	(*a*) Machined from the cast. (*b*) Annealed 7 h at 502 °K. (*c*) Permanently strained 3.8% at room temperature.	25.0	3.02
Palladium	0.9991	—	(*a*) Machined from the cast and permanently strained 1% at room temperature. (*b*) Annealed 6 h at 423 °K. (*c*) Permanently strained 6.9% at room temperature.	25.2	2.95
Platinum	0.9989	—	(*a*) Machined from the cast. (*b*) Annealed 4 h at 500°K. (*c*) Annealed 11 h at 693 °K. (*d*) Annealed 1 h at 793 °K. (*e*) Permanently strained 3.2% at room temperature.	36.0	3.00

(*) With a ring of 44.0 mm external diameter and 0.6 mm thickness.

As it is shown by the lower diagram of Fig. 1, the dissipation peak in silver is quite well marked, whilst a much lower and flatter peak was found in a previous investigation on a not very pure specimen of the same mate-

rial (5). As it was observed in lead, copper and aluminium, the temperature T_m corresponding to the maximum dissipation increases with the vibration frequency f_m of the specimen (6). The same peak is also clearly shown by the dissipation

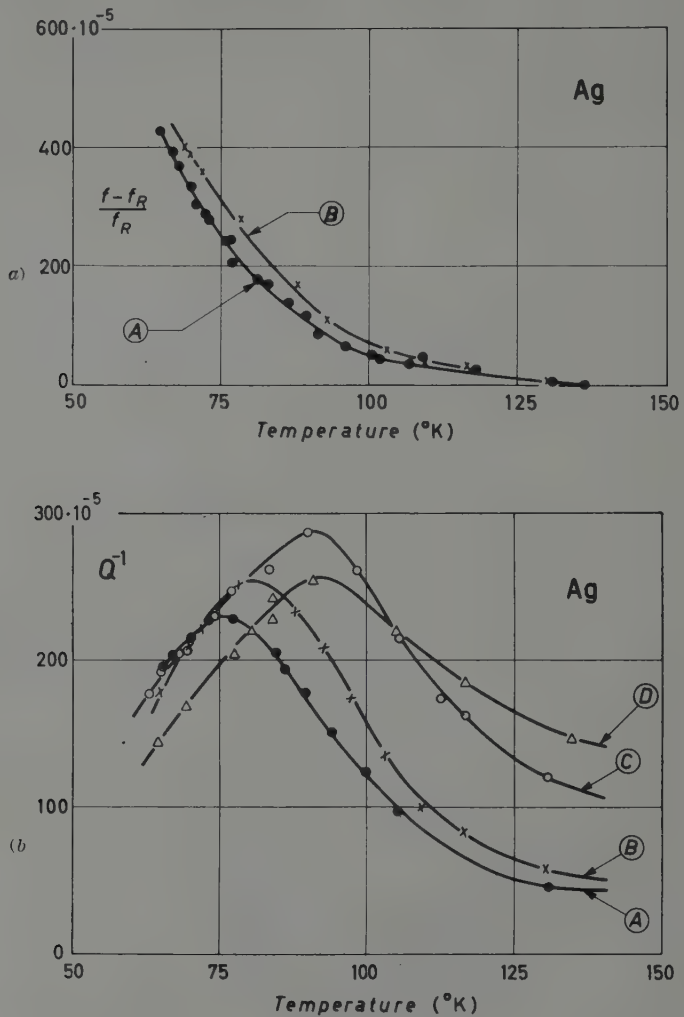

Fig. 1. – Dissipation peak and frequency relaxation in silver; curve A, $f_m=18.58$ kHz; B, $f_m=50.55$ kHz; C, $f_m=421.4$ kHz; D, $f_m=820.4$ kHz.

(5) P. G. BORDONI: *Journ. Acoust. Soc. Am.*, **26**, 495 (1954); *Ric. Scient.*, **23**, 1193 (1953).

(6) As the resonant frequency changes with temperature, f_m is the value measured at the temperature T_m.

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measurements in gold (Fig. 2, lower diagram), whilst in palladium and platinum the peak is somewhat smaller and flatter but still very evident (Fig. 3, 4 lower diagrams). As in the case of silver, the temperature T_m is an increasing function of the frequency f_m.

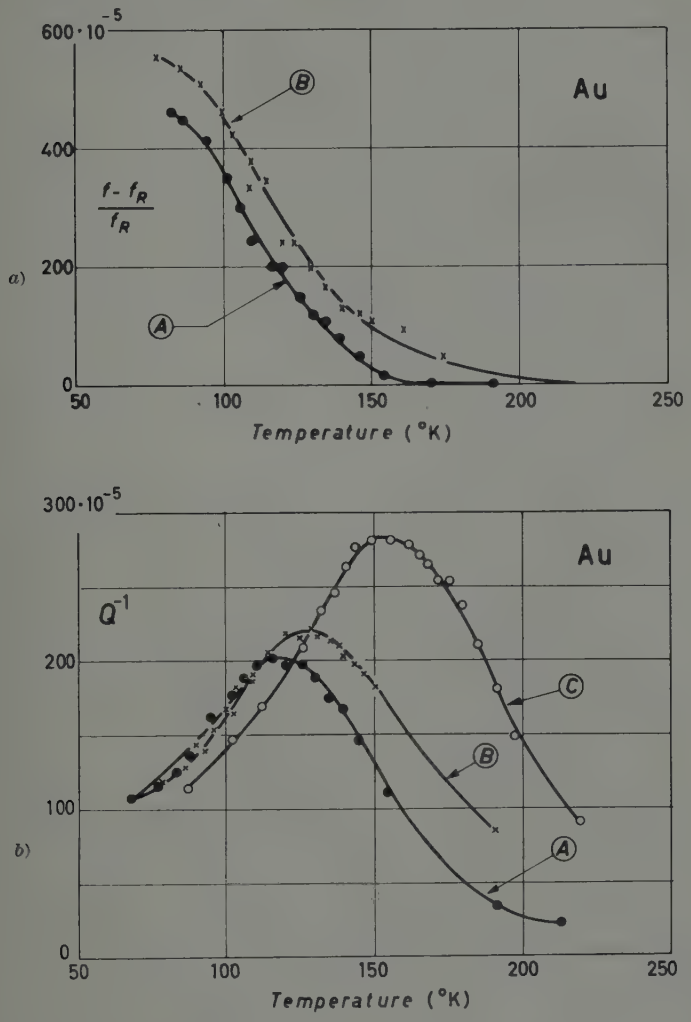


Fig. 2. – Dissipation peak and frequency relaxation in gold; curve A, $f_m = 15.99$ kHz; B, $f_m = 55.60$ kHz; C, $f_m = 561.0$ kHz.

The upper diagrams of Fig. 1-4 give the relative difference $(f - f_R)/f_R$ between the resonant frequency f measured at the temperature T, and the *relaxed* value f_R of the resonant frequency computed for the same temperature

by means of a linear extrapolation of the room temperature slope (7). These differences have been computed only for the flexural modes, as the experimental

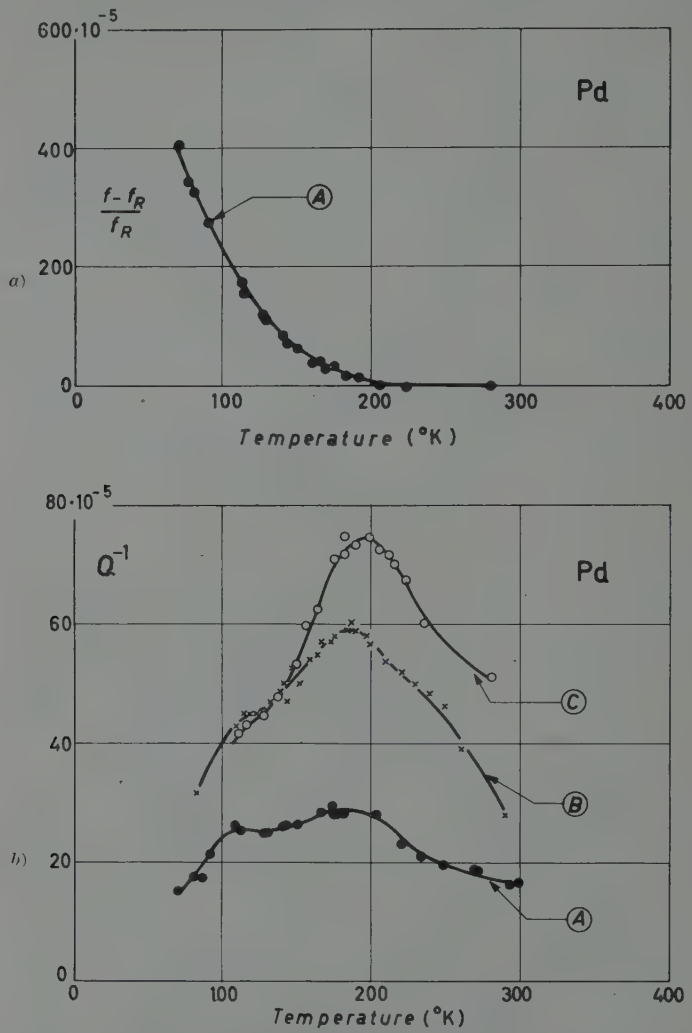


Fig. 3. - Dissipation peak and frequency relaxation in palladium; curve *A*, $f_m = 24.92$ kHz; *B*, $f_m = 88.35$ kHz; *C*, $f_m = 387.9$ kHz.

difficulties due to the grain structure of the specimens make the frequency measurements less accurate for longitudinal vibrations. In the upper diagrams of Fig. 1, 2 (silver and gold) an inflexion point is found at a temperature very

(7) See the paper quoted in footnote (2), p. 307.

near to that of the dissipation peak for the same vibration mode. This coincidence shows that the peaks are due to a *relaxation effect.*

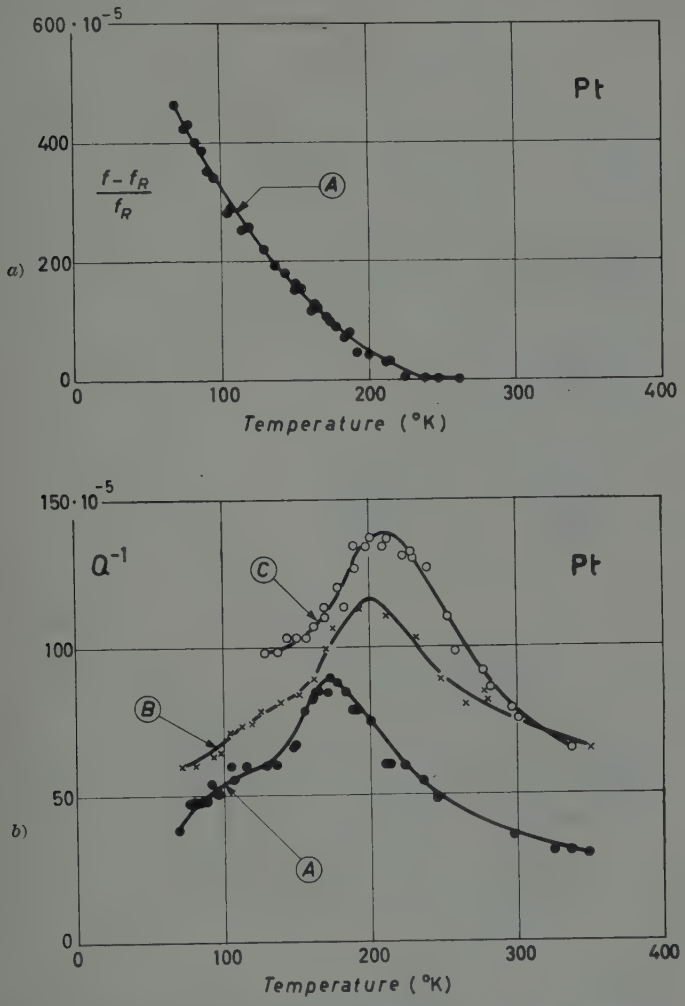


Fig. 4. – Dissipation peak and frequency relaxation in platinum; curve *A*, f_m=9.6 kHz; *B*, f_m=42.41 kHz; *C*, f_m=111.8 kHz.

The inflexion is not evident in the upper diagrams of Fig. 3, 4 (palladium and platinum) being masked by the pronounced curvature of the elastic modulus *vs.* temperature diagram exhibited by the transition metals in this temperature range ([8]). This curvature has an anomalous character as the concavity

([8]) J. A. RAYNE: *Phys. Rev.*, **118**, 1545 (1960).

is turned towards the high values of the modulus, and it is not related to any relaxation effect but to the conduction electrons. However there can be no doubt that a frequency relaxation takes place also for palladium and platinum in the same temperature interval in which the dissipation peak is observed.

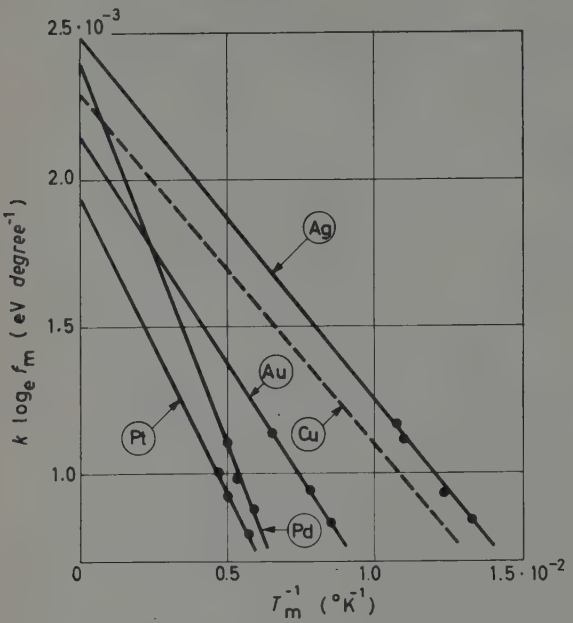


Fig. 5. – Frequency dependence of the temperature T_m of maximum dissipation.

The experimental values of f_m and T_m are listed in Table II together with the maximum energy dissipation Q_m^{-1}. When the logarithm of f_m is plotted against T_m^{-1}, as it is done in Fig. 5, the points for each metal lie on a straight line; hence the relaxation effect is *thermally activated*, as it has already been found in the case of lead, copper and aluminium.

To establish an experimental relation between the relaxation effect and the motion of dislocations, the influence of anneal and cold work on the dissipation peaks and on frequency relaxation has been systematically investigated.

It has been found that a moderate anneal decreases the height of the peak, as is shown for silver and platinum (Fig. 6, 9, lower diagrams) by the comparison of the curves *A*, obtained when the material was in the *original state* (9), and the curves *B*, taken after an annealing treatment (10).

If the temperature of the treatment is high enough, the peak may eventually be cancelled as it is shown by the curve *C* of Fig. 6 (lower diagram) for silver and by the curves *B* of Fig. 7, 8 (lower diagrams) for gold and palladium. In the case of platinum the height of the peak is also dependent upon the temperature of anneal, but some trace of it is still present after the specimen has been kept for 1 h at 793 °K (Fig. 9, lower diagram, curves *C* and *D*). It may be added that the effect of an high-temperature anneal is somewhat

(9) That is immediately after the mechanical treatments listed in Table I under (*a*) for gold, palladium and platinum, and after (*a*) and (*b*) for silver.

(10) For the details of these treatments see Table I and the captions of the figures.

TABLE II. – *Experimental data on the relaxation peak.*

Metal	Specimen	Last (*) treatment	f_m (kHz)	T_m (°K)	Q_m^{-1}
Silver	1	*a*	28.61	73.3	$129 \cdot 10^{-5}$
	2	*a*	79.99	79.3	150
	3	*b*	18.58	75.8	228
		b	50.55	81.0	254
		b	421.4	91.0	288
		b	820.4	93.0	255
		c	55.28	72.5	235
		d	—	—	—
		e	55.35	80.0	184
Gold	—	*a*	15.99	117.7	202
		a	55.60	127.7	221
		a	561.0	153.0	284
		b	—	—	—
		c	15.36	118.0	150
Palladium	—	*a*	24.92	180.0	30
		a	88.35	186.5	59
		a	387.9	200.0	75
		b	—	—	—
		c	21.69	187.0	125
Platinum	—	*a*	9.600	173.2	90
		a	42.41	201.0	116
		a	111.8	213.0	138
		b	9.641	166.0	21.5
		c	9.652	166.0	10.0
		d	9.653	166.0	9.0
		e	8.778	180.0	137

(*) For the details of thermal and mechanical treatments, see Table I.

masked in silver by the large increase of the dissipation measured at room temperature which has already been observed in copper after a thermal

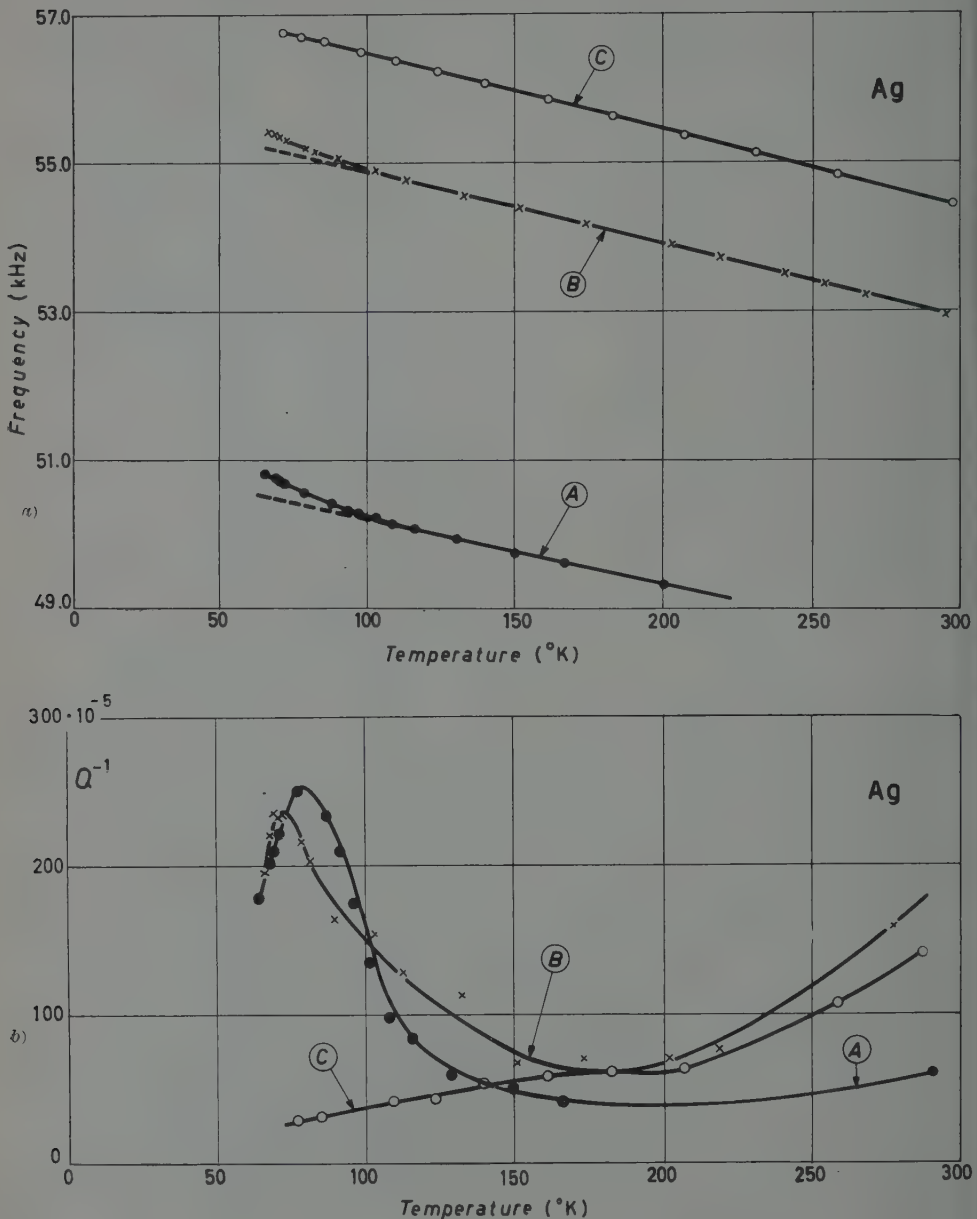
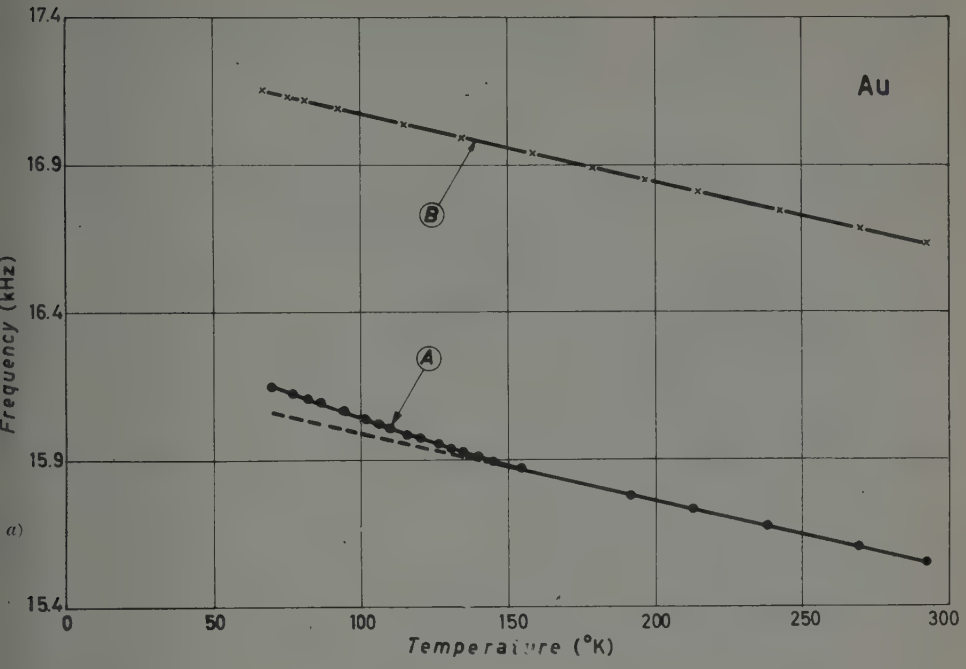
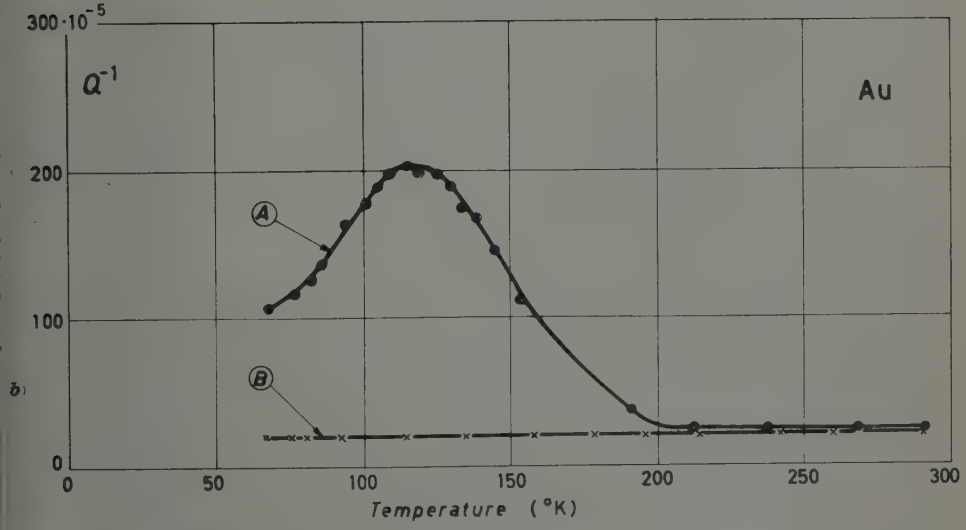


Fig. 6. – Influence of anneal on the height of the dissipation peak and on the frequency relaxation in silver. Curve *A*, after the treatments (*a*) and (*b*) of Table I; *B*, after 3 h at 460 °K; *C*, after 4 h at 493 °K.



a)



b)

Fig. 7. – Influence of anneal on the height of the dissipation peak and on the frequency relaxation in gold. Curve *A*, after the treatment (*a*) of Table I; *B*, after 7 h at 502 °K.

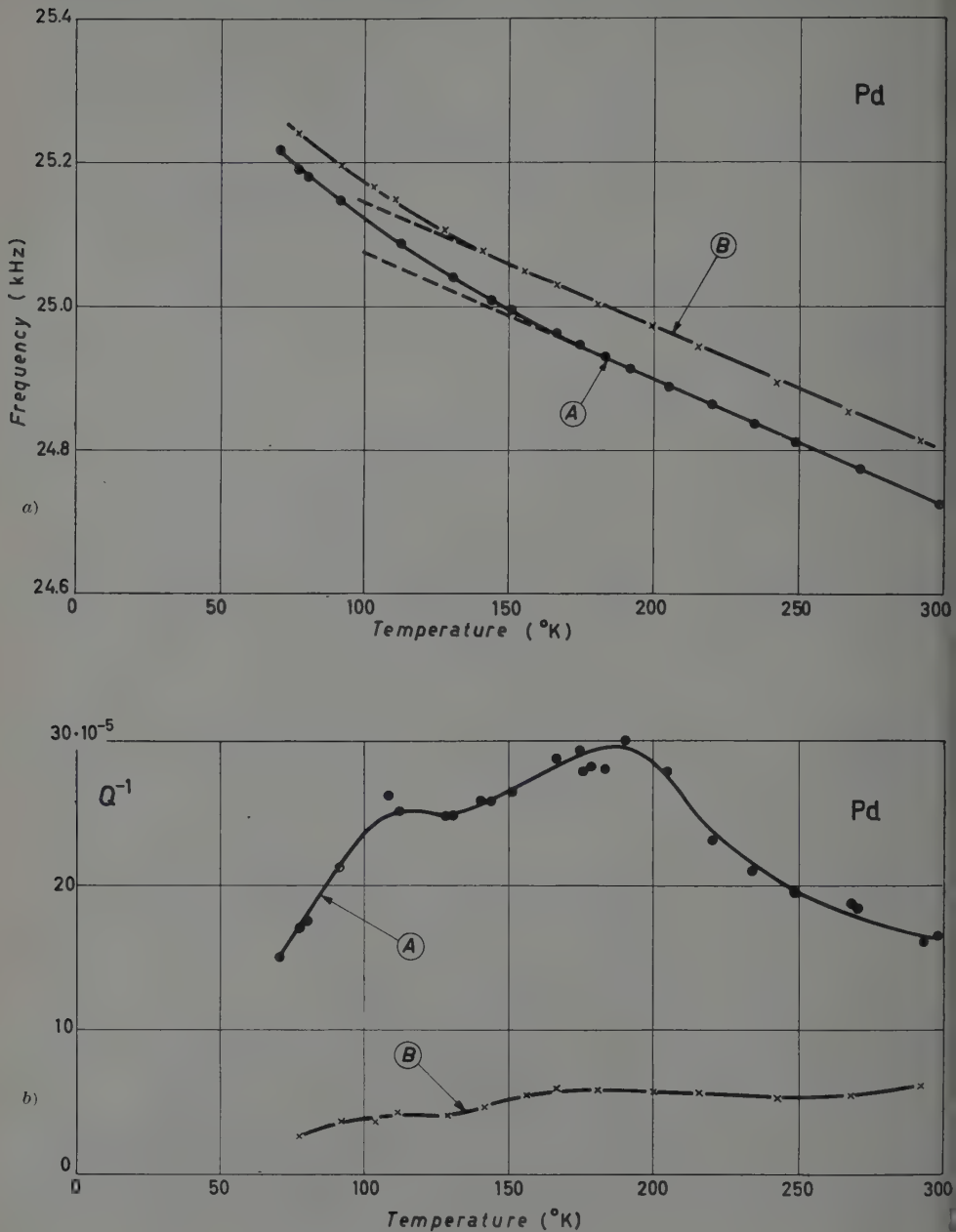


Fig. 8. – Influence of anneal on the height of the dissipation peak and on the frequency relaxation in palladium. Curve *A*, after the treatment (*a*) of Table I; *B*, after 6 h at 423 °K.

treatment ([11]) and which is not related to the relaxation effect. No such increase is noticed in the other metals, which seem therefore to be better

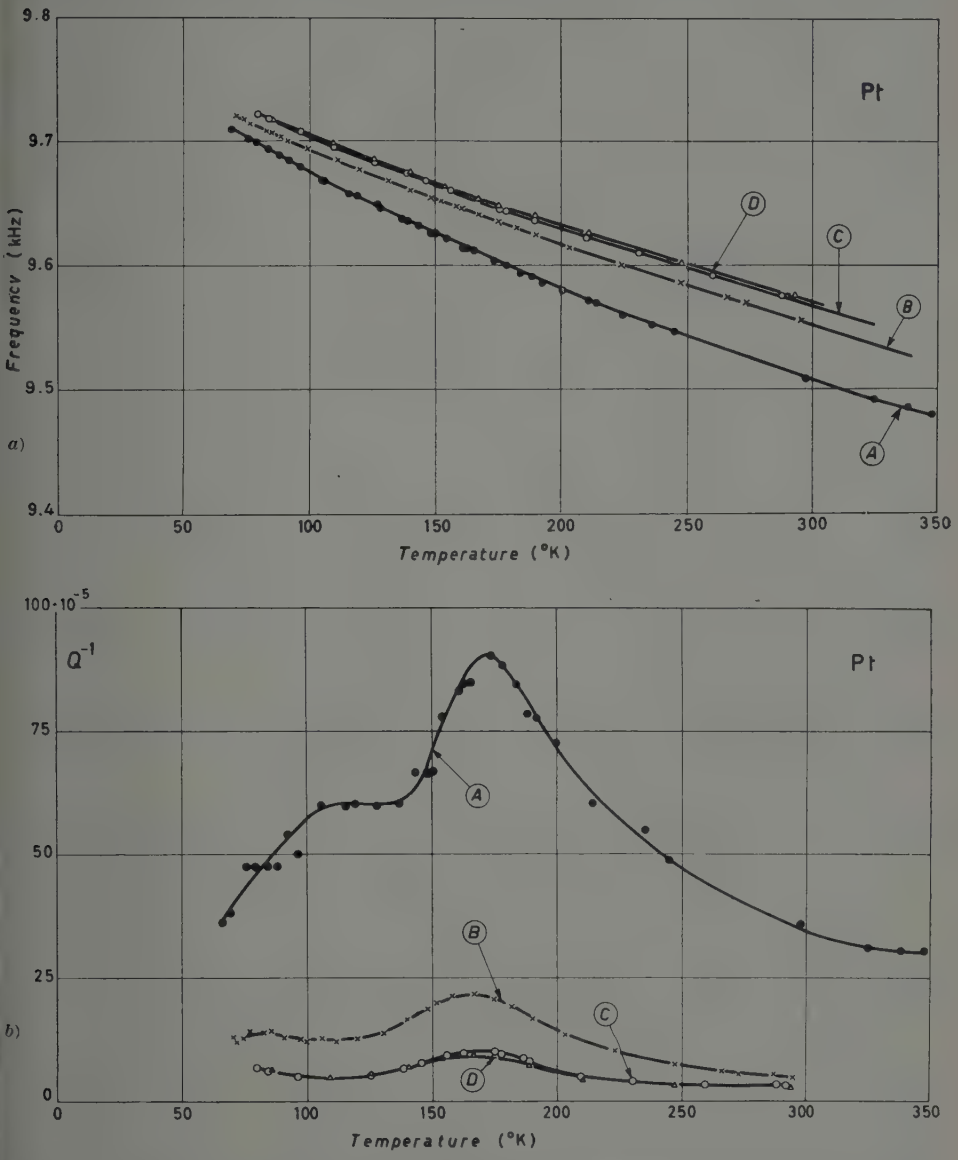


Fig. 9. – Influence of anneal on the height of the dissipation peak and on the frequency relaxation in platinum. Curve *A*, after the treatment (*a*) on Table I; *B*, after 4 h at 500 °K; *C*, after 11 h at 693 °K; *D*, after 1 h at 793 °K.

([11]) See the paper quoted in footnote ([2]), p. 292, or D. H. Niblett and J. Wilks: *Phil. Mag.*, **1**, 415 (1956); **2**, 1427 (1957); *Adv. in Phys.*, **9**, 1 (1960).

suited for an investigation on the influence of anneal upon the dissipation peak.

All the annealing treatments were made in an evacuated container and they lasted long enough to allow an equilibrium state to be reached by the material. One of the resonant frequencies of the specimen was measured during the treatment, and the anneal was not stopped until the frequency was approximately constant with time, as it is shown for silver and gold, in Fig. 10.

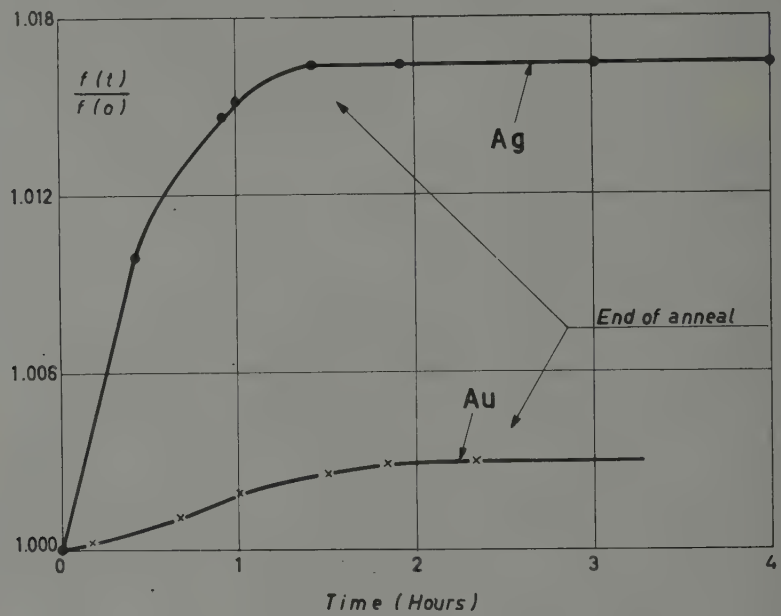


Fig. 10. – Increase of frequency $f(t)$ with time t at the constant temperature of 493 °K for silver and of 502 °K for gold.

The upper diagrams of Fig. 6-9 show the influence of anneal upon the frequency of the lowest flexural mode of the specimens. For silver and gold (Fig. 6, 7, upper diagrams) this influence is similar to that already observed in copper ([12]). The frequency *vs.* temperature curve is shifted after anneal towards higher values of the frequency. Its inflexion decreases with the height of the dissipation peak, as it is required by the relaxation theory, whilst its slope is little or not changed at temperatures higher than T_m. After an high-temperature anneal, when the dissipation peak has been cancelled, the frequency curve no longer exhibits any trace of inflexion.

([12]) See the paper quoted in footnote ([2]), p. 291.

In palladium and platinum (Fig. 8, 9, upper diagrams) the effect of anneal on the resonant frequency is less obvious, as the frequency *vs.* temperature

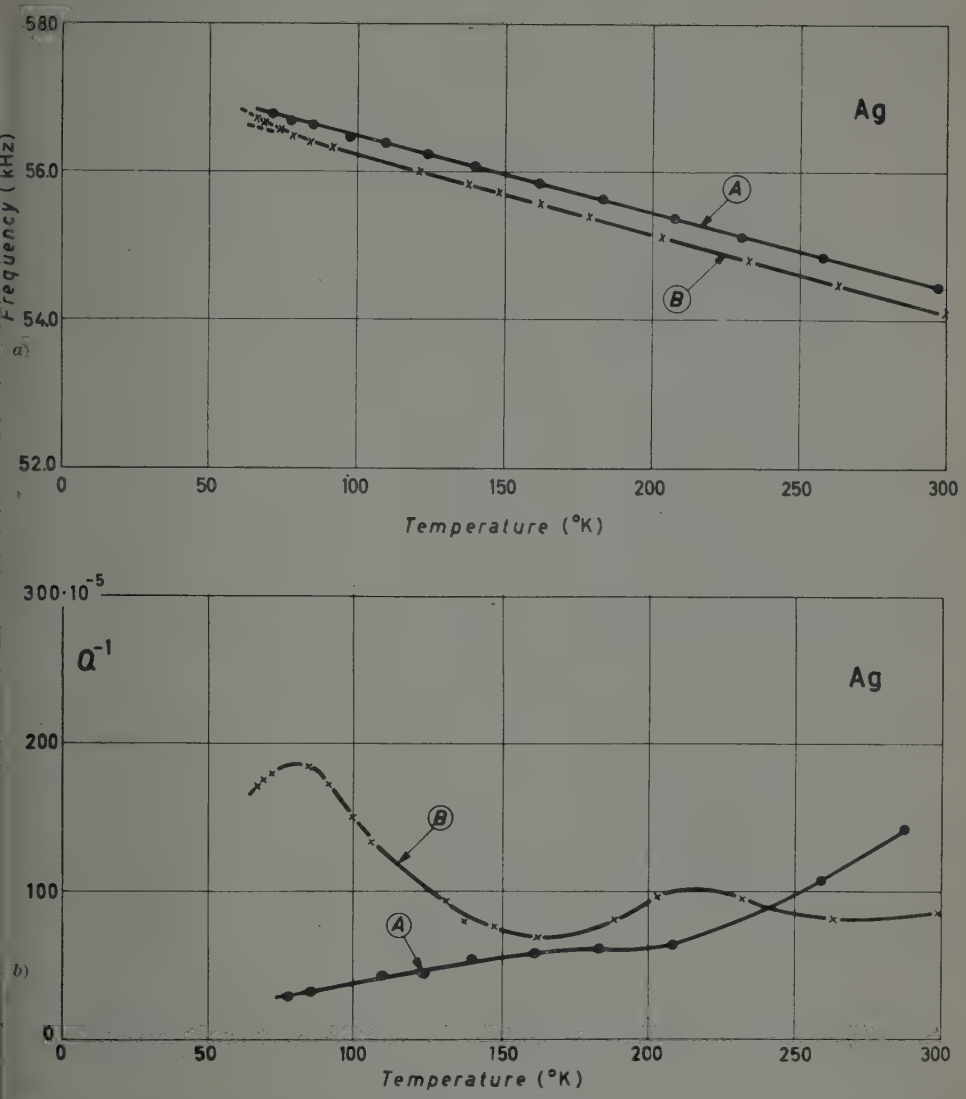


Fig. 11. – Influence of cold work on the height of the dissipation peak and on the frequency relaxation in silver. Curve *A*, after an anneal of 4 h at 493 °K; *B*, after 2.7% permanent axial strain.

curve, even after a complete anneal, is a curved line turning its concavity towards the high frequencies, in agreement with the previous remark on the elastic modulus of the transition elements. However a slight reduction of the

curvature is observed after each annealing treatment and this is what one would expect from the relaxation theory.

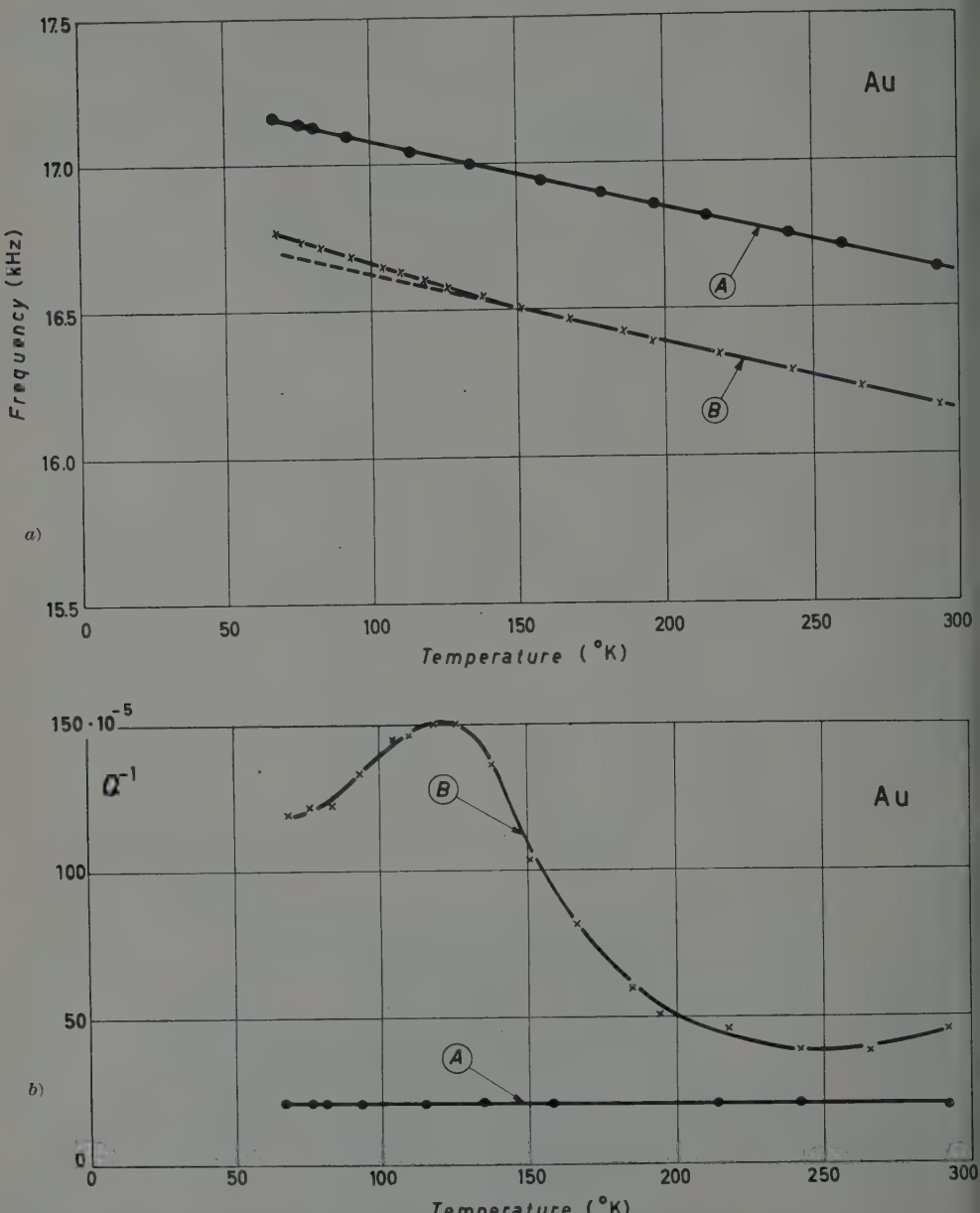


Fig. 12. – Influence of cold work on the height of the dissipation peak and on the frequency relaxation in gold. Curve *A*, after an anneal of 7 h at 502 °K; *B*, after 3.8% permanent axial strain.

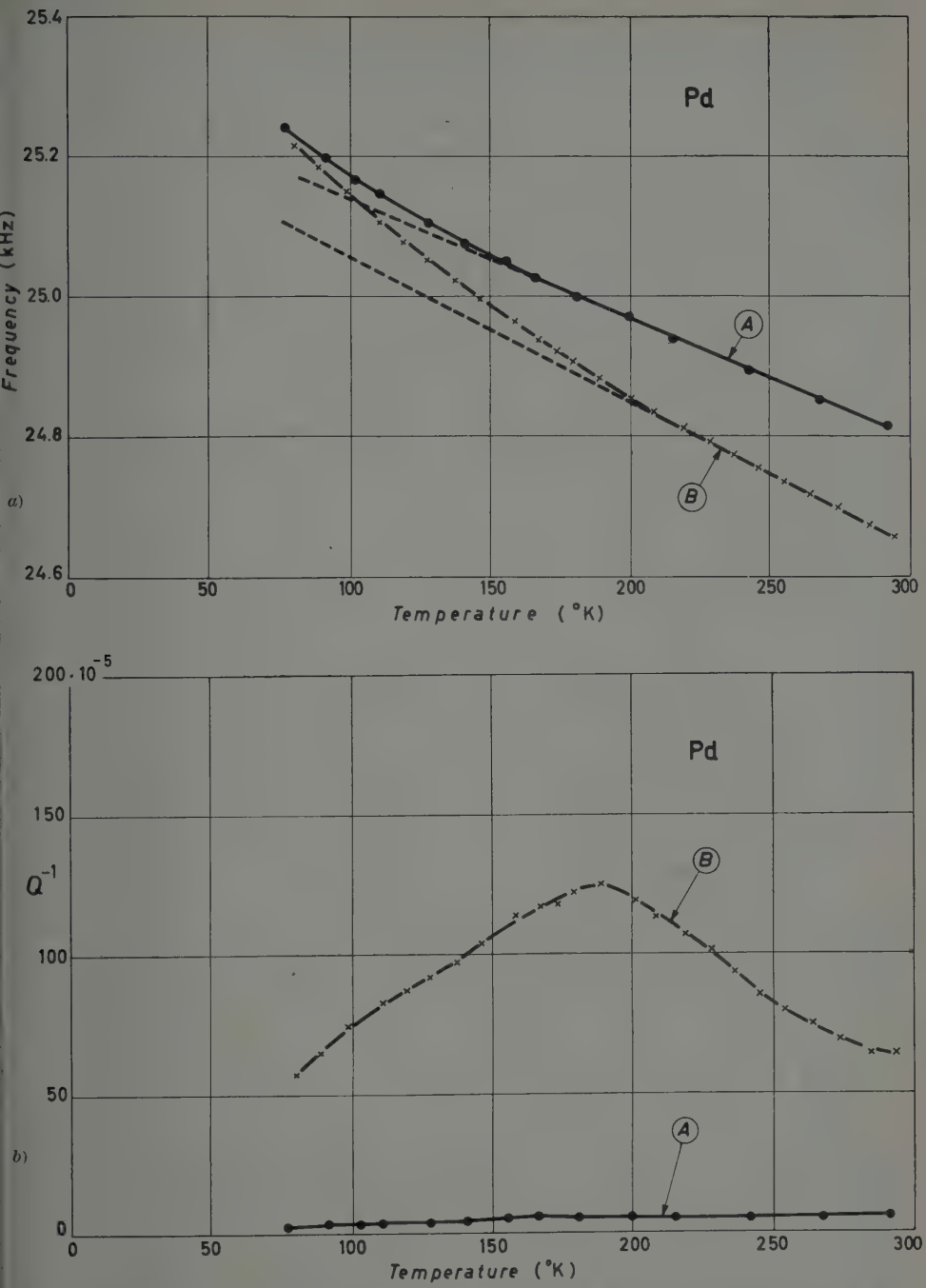


Fig. 13. – Influence of cold work on the height of the dissipation peak and on the frequency relaxation in palladium. Curve *A*, after an anneal of 6 h at 423 °K; *B*, after 6.9% permanent axial strain.

The influence of cold work is the opposite of that of anneal. If the peak has been cancelled by a high temperature treatment, as it is the case with

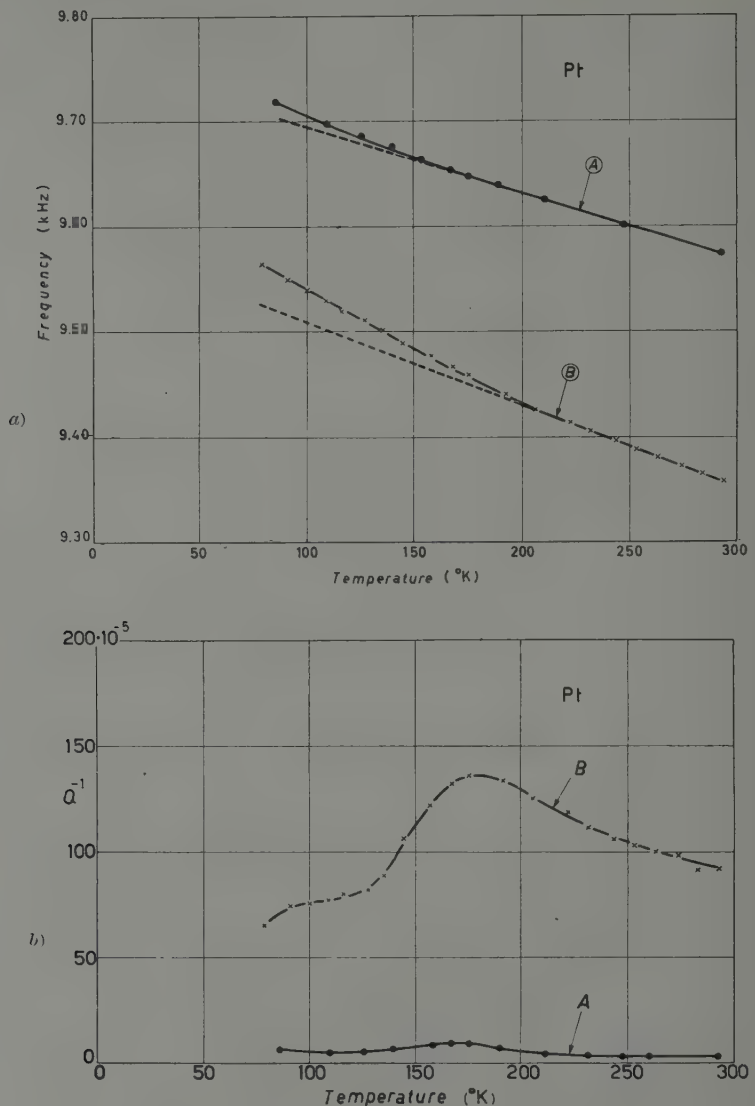


Fig. 14. – Influence of cold work on the height of the dissipation peak and on the frequency relaxation in platinum. Curve *A*, after an anneal of 1 h at 793 °K; *B*, after 3.2% permanent axial strain.

silver after the treatment (*d*) of Table I, a permanent strain of 2.7% due to an axial compression of the plate at room temperature reintroduces the dis-

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sipation maximum (Fig. 11, lower diagram, curve *B*). At the same time an inflexion appears in the frequency *vs.* temperature curve (Fig. 11, upper diagram, curve *B*). The same influence is observed in gold, when it is permanently strained 3.8% after a complete anneal (Fig. 12) and in palladium after a permanent strain of 6.9% (Fig. 13). In the last case the peak is even higher than in the *original state* of the material corresponding to a permanent strain of only 1% (Table I).

As it has already been said, even an annealing treatment at 793 °K does not remove completely the peak in platinum, but reduces it to about one tenth of its original value. This small peak is increased again to nearly twice its original value by a permanent strain of 3.2% (Fig. 14). Hence the influence of cold work is the same as in the other metals.

All the previous measurements have been made on polycrystalline specimens. However the dissipation peak cannot be due to the grain structure, owing to its low temperature and to the fact that in copper the *same* peak has been observed by various experimenters in single crystals and in the polycrystalline material ([13]).

The opposite influences of anneal and cold work point out that the relaxation effect is due to some geometrical imperfections of the crystal lattice. As it has been suggested by SEEGER ([14]) the comparatively high value of the annealing temperature required to cancel the peak excludes that the relaxation effect can be due to the reorientation of divacancies or similar lattice defects created by the plastic deformations as these defects anneal out rapidly at room temperature, and indicates that the peaks are produced by the motion of dislocation segments near their equilibrium positions.

3. – Discussion.

3·1. *Activation energy.* – The straight lines of Fig. 5 show that the vibration frequency f_m is related to the temperature T_m at which the dissipation peak is observed by an Arrhenius equation

$$f_m = f_{m,0} \exp\left[-\frac{\overline{W}}{kT_m}\right], \tag{1}$$

where $\overline{W}$ is obtained from the slope of the lines, and characterizes the activation energy *associated with the displacement of the peak*; $f_{m,0}$ is obtained from

([13]) See P. G. BORDONI: *Journ. Acoust. Soc. Am.*, **26**, 495 (1954); *Ric. Scient.*, **23**, 1193 (1953); H. L. CASWELL: *Journ. Appl. Phys.*, **29**, 1210 (1958); V. K. PARÈ: Thesis Cornell University, AFOSR-TR 58-92.

([14]) A. SEEGER: *Phil. Mag.*, **1**, 651 (1956).

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the intercept of the straight lines with the frequency axis and gives the limiting value of the frequency at which the peak would be observed for an infinite value of the temperature T_m.

From a purely formal stand point a new parameter, the characteristic time $\bar{\tau}$, may be substituted to f_m by means of the relation

$$2\pi f_m \bar{\tau} = 1 \,. \tag{2}$$

The temperature dependence of $\bar{\tau}$ is obtained from (1) and (2)

$$\bar{\tau} = \bar{\tau}_0 \exp\left[\frac{\overline{W}}{kT_m}\right], \tag{3}$$

where $\bar{\tau}_0$ is the value given by (2) when $f_{m,0}$ is substituted to f_m.

The values of $\overline{W}$ and $\bar{\tau}_0$ obtained from the experimental data by means of (2) and (3) are listed in Table III. For comparison purposes the values obtained for the same parameters in previous investigations on copper ([15])

TABLE III. – *Fundamental parameters of the relaxation effect.*

Metal	Last (*) treatment	$\overline{W}$ (eV)	$\bar{\tau}_0^{-1}$ (s^{-1})	Debye's frequency ν_D (s^{-1})	σ_p^0/μ	$\frac{\bar{\tau}_0 c_t \overline{W}}{b^4 \mu}$
Silver (**)	(*b*)	0.124 ($\pm$0.005)	$40 \cdot 10^{11}$	$45 \cdot 10^{11}$	$6.0 \cdot 10^{-4}$	0.04
Gold	(*a*)	0.158 ($\pm$0.002)	0.7	35	8.6	2.2
Palladium	(*a*)	0.260 ($\pm$0.013)	12.0	57	10.3	0.3
Platinum	(*a*)	0.192 ($\pm$0.006)	0.06	47	3.2	23.0
Copper (**)	—	0.122 ($\pm$0.005)	3.8	67	4.2	0.6
Aluminium (**)	—	0.11	0.13	82	5.0	24.0

(*) For the details of thermal and mechanical treatments, see Table I.
(**) Measurements made on the specimen no. 3.
(**) See the paper quoted in footnote ([2]), p. 296.
(**) See the paper quoted in footnote ([16]).

([15]) See the paper quoted in footnote ([2]), p. 297.

and aluminium ([16]) are also listed at the end of the Table. It is worth of notice that the activation energies $\overline{W}$ for the different metals are rather near each other, the largest value being that measured in palladium (0.260 eV), and the smallest that found in aluminium ([17]) (0.11 eV). This similarity shows that $\overline{W}$ is associated with some intrinsic property of the dislocations and not with their density or with the impurity content.

The inverse $\overline{\tau}_0^{-1}$ of the limiting value of the characteristic time is rather close to the Debye's frequency ν_D computed by means of the simple equation

$$\nu_D = \frac{k\Theta_D}{h}, \tag{4}$$

and given in the fourth column of Table III ([18]).

The experimental values of $\overline{W}$ and $\overline{\tau}_0^{-1}$ may be employed to test the validity of the present theories of dislocation motion. As it has been shown in a recent analysis ([19]) the fundamental hypotheses adopted by WEERTMAN ([20]), SEEGER ([21]) and by DONTH ([22]) are rather different, but the final formulae for the numerical computation of the potential barrier H which opposes the motion of dislocations and of the frequency ν associated with this motion ([23]) are surprisingly similar.

In Weertman's theory the value H_W of the potential barrier is given by

$$H_W = \frac{2}{\pi}\mu b^3 \left(\frac{\sigma_p^0}{\mu}\right)^{\frac{1}{2}}, \tag{5}$$

where b is the length of Burger's vector; μ is the torsional modulus; σ_p^0 is the Peierls' stress for the limiting case of a vanishing thermal agitation.

In Seeger's theory the potential barrier H_S depends upon the *stress* pro-

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([16]) N. G. EINSPRUCH and R. TRUELL: *Phys. Rev.*, **109**, 652 (1957).

([17]) An even smaller value of $\overline{W}$ seems to be characteristic of lead. See W. P. MASON: *Physical Acoustics and the Properties of Solids* (New York, 1958), p. 267. However the experimental evidence for this metal is still too limited to allow any conclusions to be drawn safely on the value of $\overline{W}$.

([18]) A more refined computation of the Debye's frequencies associated with the different types of waves is obviously not necessary as the comparison between $\overline{\tau}_0^{-1}$ and ν_0 has only a qualitative meaning.

([19]) P. G. BORDONI: *Dislocation Relaxation at High Frequencies*, in *Suppl. Nuovo Cimento*, vol. XVII, pag. 43 (1960).

([20]) J. WEERTMAN: *Journ. Appl. Phys.*, **26**, 202 (1955).

([21]) See the paper quoted in footnote ([14]).

([22]) H. DONTH: *Zeits. f. Phys.*, **149**, 111 (1957). See also A. SEEGER, H. DONTH and P. PFAFF: *Discussions Faraday Society*, **23**, 19 (1957).

([23]) In this case the word « *frequency* » has only a statistical meaning.

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duced by the vibrations. However, when the *strain* is from 10^{-8} to 10^{-6} as in the present measurements, a satisfactory average of H_S is given by the very simple relation

$$H_S = 1.25\, H_W \,. \tag{6}$$

In the more recent theory due to SEEGER, DONTH and PFAFF ([22]), the relation between ν and the temperature is *not* exactly of the exponential type. As the temperature interval in which the peaks are observed is rather limited, the temperature dependence of the dislocation frequency ν can be represented with a satisfactory approximation by an Arrhenius equation, with an activation energy H_{SDP} given by

$$H_{SDP} = 1.4\, H_W \,. \tag{6'}$$

The limiting value ν_0 assumed by the frequency associated with the dislocation motion for very high values of T has the following expression according to SEEGER

$$(\nu_0)_S = \frac{c_t}{b\sqrt{2\pi}} \left(\frac{\sigma_p^0}{\mu}\right)^{\frac{1}{2}}, \tag{7}$$

where c_t is the velocity of transversal waves. In the S.D.P. theory the coefficients $(\nu_0)_{SDP}$ of the exponential depends upon the temperature according to the equation

$$(\nu_0)_{SDP} = \pi^3 \frac{\sqrt{5}}{16}\, e\, \frac{c_t k T}{b^4 \mu} \,, \tag{7'}$$

where e is the base of logarithms and k is the Boltzmann's constant. However it must be remembered that the parameter given by the experiment is not the frequency $f_{m\,0}$ but its logarithm. Hence in the narrow interval of temperatures in which the peaks are observed the temperature dependence of $(\nu_0)_{SDP}$ can be neglected, taking for T an intermediate value between those listed in Table II for T_m ([24]). When this is done the approximate values of $(\nu_0)_{SDP}$ are near to those given by (7).

A first check of the theory may be obtained assuming with SEEGER that the value H_S of the potential barrier equals the activation energy, that is

$$H_S = \overline{W} \,, \tag{8}$$

([24]) The relative error introduced in this way in the computation of $\ln (\nu_0)_{SDP}$ does not exceed $\pm 6\%$. For the derivation of (7') see the paper quoted in footnote ([19]).

and employing the experimental data for $\overline{W}$ given in Table III to compute the ratio of the Peierls' stress σ_p^0 to the torsional modulus μ by means of (5) and (6).

The values obtained are listed in the sixth column of Table III and are of the order of a few units in 10^{-4}, that is somewhat larger than those obtained in static experiments. This has been explained by SEEGER ([25]) observing that a great portion of dislocations in a crystal does not lie along close-packed directions. Therefore their effective Peierls' stress is several orders of magnitude smaller than σ_p^0, and, even at low temperatures, the crystal deforms at stresses smaller than σ_p^0. Only if all dislocations involved in an experiment lie along a crystallographic direction it is possible to observe the Peierls' stress in static experiments.

Another significant test of the theory can be made by eliminating (σ_0^p/μ) between (5), (6) and (7) and observing that the ratio $(c_t H_S)/[2\pi(\nu_0)_S b^4 \mu]$ is an invariant for all metals. If we make the reasonable assumption that the unknown ratio between the theoretical value of $(\nu_0)_S$ and the experimental value of $\bar{\tau}_0^{-1}$ has the same value for all metals having the same structure ([26]), the above expression must also be invariant when the experimental values of $\overline{W}$ and $\bar{\tau}_0^{-1}$ are substituted to H_S and $2\pi(\nu_0)_S$. The last column of Table III shows that this is not the case. As the hypothesis (8) cannot be reasonably questioned, it seems that the lack of agreement between theory and experiment is essentially due to the evaluation of $(\nu_0)_S$ which is the less satisfactory part of the present theories of dislocation motion.

3·2. *Relaxation spectra.* – It has been shown elsewhere ([27]) that all the experimental points obtained in energy dissipation measurements made at different frequencies must fall on a single line when the ratio Q^{-1}/Q_m^{-1} is plotted against $\overline{W}k^{-1}(T_m^{-1} - T^{-1})$, provided the relaxation effect is due to the sum of many elementary effects having the same activation energy $\overline{W}$. Fig. 15-18 show that this is the case for the four metals; it is also evident that the experimental peaks are much broader than the theoretical curve for a single relaxation time (dotted line), and that the low-temperature branch is in every case higher than the high-temperature one. These results are qualitatively the same which have been obtained for copper, and they show that the dissipation peak is associated with a spectrum of relaxation times. The

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([25]) See the paper quoted in footnote ([14]). The value of σ_p^0/μ would essentially be the same if H_W or H_{BDP} were taken instead of H_S.

([26]) SEEGER takes $(\nu_0)_S = \bar{\tau}_0^{-1}/(2\pi)$.

([27]) See the paper quoted in footnote ([2]), p. 300, or P. G. BORDONI: *Theory of relaxation effects with a continuous spectrum*, in *Proc. of the Third International Congress of Acoustics* (Stuttgart, September 1959), in press.

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activation energy W associated with each time τ has essentially the same value $\overline{W}$ which characterizes the frequency dependence of the temperature of the dissipation peak.

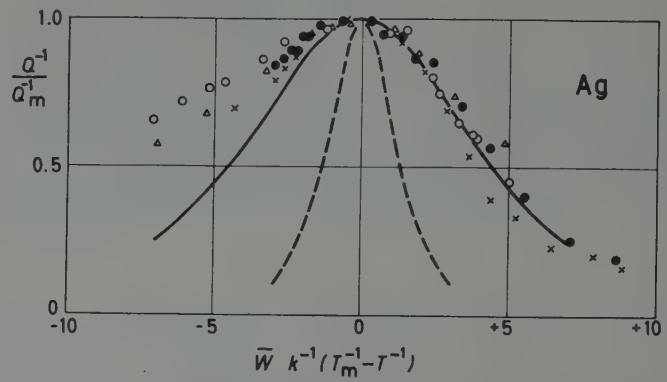

Fig. 15. – Values of the ratio Q^{-1}/Q_m^{-1} as a function of $\overline{W}k^{-1}(T_m^{-1} - T^{-1})$ for silver, after the treatments (*a*) and (*b*). Measurements at: (●) 19 kHz; (×) 51 kHz; (○) 421 kHz; (△) 820 kHz. Heavy line, sech $[0.27\ \overline{W}k^{-1}(T_m^{-1} - T^{-1})]$; dotted line, Zener's dissipation curve for a single relaxation time.

The symmetrical part of the dissipation curve, plotted as a function of $\overline{W}k^{-1}(T_m^{-1} - T^{-1})$, is due to a symmetrical spectrum of relaxation times, whose

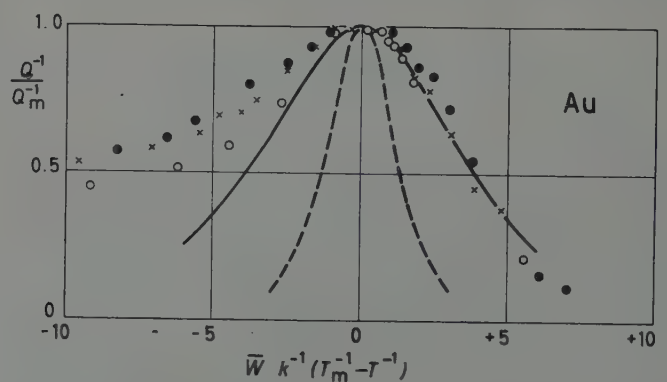

Fig. 16. – Values of the ratio Q^{-1}/Q_m^{-1} as a function of $\overline{W}k^{-1}(T_m^{-1} - T^{-1})$ for gold, after the treatment (*a*). Measurements at: (●) 16 kHz; (×) 56 kHz; (○) 561 kHz. Heavy line, sech $[0.35\ \overline{W}k^{-1}(T_m^{-1} - T^{-1})]$; dotted line, Zener's dissipation curve for a single relaxation time.

center coincides with the characteristic time $\overline{\tau}$ which is directly related through eq. (1) and (2) to the frequency dependence of the dissipation peak.

The width of the spectrum can be evaluated with good approximation by applying to the high temperature branch of the experimental curves, which

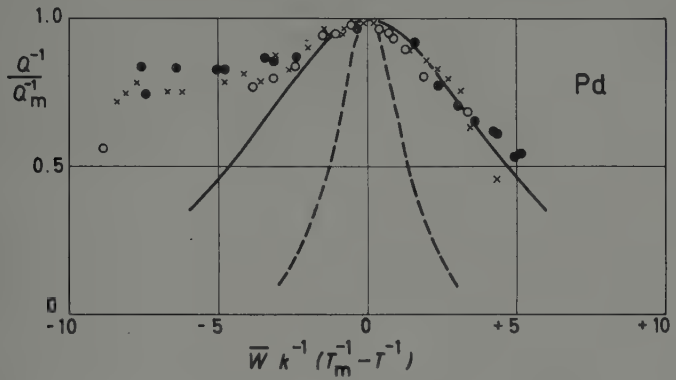

Fig. 17. – Values of the ratio Q^{-1}/Q_m^{-1} as a function of $\overline{W}k^{-1}(T_m^{-1} - T^{-1})$ for palladium, after the treatment (*a*). Measurements at: (●) 25 kHz; (⁄) 88 kHz; (○) 388 kHz. Heavy line, sech $[0.22\,\overline{W}k^{-1}(T_m^{-1} - T^{-1})]$; dotted line, Zener's dissipation curve for a single relaxation time.

is less affected by the asymmetry of the spectrum, the Fuoss-Kirkwood equation ([28])

$$\frac{Q^{-1}}{Q_m^{-1}} = \operatorname{sech}\,[\gamma\overline{W}k^{-1}(T_m^{-1} - T^{-1})]\,, \tag{9}$$

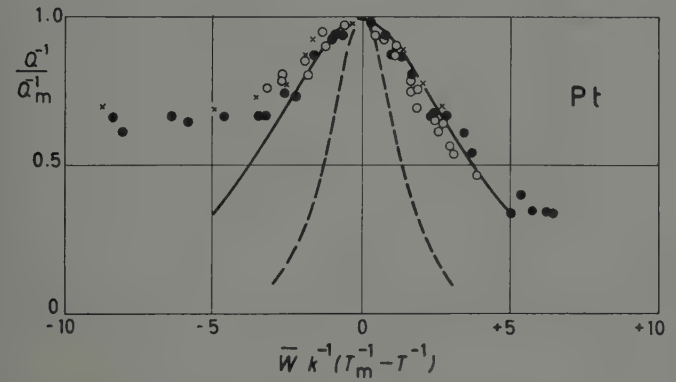

Fig. 18. – Values of the ratio Q^{-1}/Q_m^{-1} as a function of $\overline{W}k^{-1}(T_m^{-1} - T^{-1})$ for platinum, after the treatment (*a*). Measurements at: (●) 10 kHz; (×) 42 kHz; (○) 112 kHz. Heavy line, sech $[0.35\,\overline{W}k^{-1}(T_m^{-1} - T^{-1})]$; dotted line, Zener's dissipation curve for a single relaxation time.

([28]) R. M. Fuoss and J. G. Kirkwood: *Journ. Chem. Phys.*, **63**, 385 (1941). For additional information on this approximation of the spectrum see also the paper quoted in footnote ([2]), p. 305.

where γ is a parameter which characterizes the spectrum width; its value is unity for a single relaxation time, whilst it vanishes for an infinite spectrum width. According to the theory developed by Fuoss and Kirkwood, the density $\delta(\tau)$ of the relaxation spectrum is a bell-shaped curve, having its maximum

TABLE IV. – *Relaxation spectra.*

Metal	Last treatment (*)	γ	$\ln \tau_2/\tau_1$
Silver (**)	(*a*)	0.28	5.30
Silver	(*b*)	0.27	5.60
	(*c*)	0.30	4.87
	(*e*)	0.18 (*,*)	9.05
Gold	(*a*)	0.35	3.91
	(*c*)	0.27 (*,*)	5.60
Palladium	(*a*)	0.22	7.38
	(*c*)	0.24 (*,*)	6.55
Platinum	(*a*)	0.35	3.91
	(*b*)	0.35	3.91
	(*c*)	0.35	3.91
	(*d*)	0.35	3.91
	(*e*)	0.24 (*,*)	6.55
Copper	—	0.39 (*,*)	2.89

(*) For the details of thermal and mechanical treatments, see Table I.
(**) Measurements made on specimens no. 1 and 2.
(*,*) Computed with the value of $\overline{W}$ given by Table III and measured before cold work.
(*,*) See the paper quoted in footnote (3), p. 309.

at $\bar{\tau}$; the times τ_2 and τ_1 for which the height of the spectrum is $1/\sqrt{2}$ of the maximum height may be taken as upper and lower limits of the spectrum; the logarithm of their ratio gives a physically significant measure of the spectrum width.

The heavy lines of Fig. 15-18, which have been computed by means of (9),

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fit well the experimental points for $T \geqslant T_m$, showing that the Fuoss-Kirkwood theory gives a good approximation of the symmetrical part of the spectrum. The values of γ and of $\ln(\tau_2/\tau_1)$ have been computed for all the measurements made, and the values obtained are listed in Table IV.

The widths of the spectra are somewhat larger than the value previously found in copper which is given for comparison purposes at the end of the same Table. It may be noticed that anneal decreases the width, whilst the opposite effect is produced by cold work.

3·3. *Frequency relaxation.* – As it has been said, the experimental values of $(f-f_R)/f_R$ are available only for the lowest vibration modes (Fig. 1-4, upper diagrams).

For silver and gold the agreement between theory and experiment is satisfactory. According to the Fuoss-Kirkwood's computation, the maximum dissipation Q_m^{-1} is related to the *total relaxation* of the resonant frequency by the following equation

$$Q_m^{-1} = \gamma \frac{f_U - f_R}{f_R} = \gamma \frac{S}{2}, \tag{10}$$

where S is the *relaxation strength*, and $(f_U - f_R)$ is the value taken by the difference $(f-f_R)$ for $T \ll T_m$, that is when $\bar{\tau}$ is much longer than the vibration period ([29]).

The ratio $(f_U-f_R)/f_R$ can be obtained from the upper diagrams of Fig. 1 and 2, extrapolated towards the low temperatures. The values given by this extrapolation are in satisfactory agreement with those computed by means of equation (10) taking for Q_m^{-1} and γ the values given respectively by Tables II and IV. It must however be noticed that the inflexion point of the curve which gives $(f-f_R)/f_R$ as a function of T, is not found at the temperature T_m as theory requires, but at a somewhat lower temperature.

The above comparison between the maximum dissipation and the relaxation strength cannot be made for palladium and platinum, as the computation of f_R cannot be made by a linear extrapolation owing to the anomalous behaviour of the modulus *vs.* temperature curve which has already been pointed out.

3·4. *Influence of anneal and cold work.* – The influence of anneal on the relaxation effect, that is the reduction of the dissipation peak and of the relaxation observed in the resonant frequency (Fig. 6-9) is readily explained

([29]) The difference $(f_U - f_R)$ must be computed at temperatures lower than T_m but not too near to the absolute zero, otherwise the extrapolation which gives f_R as a linear function of T is not valid.

by a decrease of the number of dislocations which are active in the relaxation process. The results obtained throw also some light on another effect due to anneal, that is the temperature-independent increase of the frequency of the specimen, which was first observed in copper ([30]). This increase is shown by the upper diagrams of Fig. 6-9. In silver, as in copper, a contemporary increase of the room temperature dissipation is observed (Fig. 6, lower diagram). No change in dissipation is found in gold (Fig. 7, lower diagram) whilst in palladium and platinum anneal gives rise to a *decrease* of the room temperature dissipation (Fig. 8, 9, lower diagrams).

The relative values of the permanent changes observed in frequency and dissipation after anneal are listed in Table V, together with those obtained for

TABLE V. – *Permanent changes in resonant frequency and energy dissipation at room temperature after anneal.*

Metal	Specimen	Last treatment (*)	Changes in	
			Frequency ($\Delta f/f$)	Dissipation (ΔQ^{-1})
Silver	3	(*c*)	$+ 9000 \cdot 10^{-5}$	$+ 80 \cdot 10^{-5}$
		(*d*)	+11 600	+130
Gold	—	(*b*)	+ 6 800	+ 5
Palladium	—	(*b*)	+ 320	— 10
Platinum	—	(*b*)	+ 460	— 25
		(*c*)	+ 630	— 27
Copper (**)	3	1 h at 515 °K	+ 980	+ 29
	2	6 h at 516 °K	+ 2370	+130
	1 (*b*)	1 h at 873 °K	+ 2920	+120

(*) For the details of thermal and mechanical treatments, see Table I.
(**) See the paper quoted in footnote ([2]), p. 291.

copper in the previous work. In every case the changes in frequency are at least one order of magnitude larger than those measured for Q^{-1}. Moreover

([30]) See the paper quoted in footnote ([2]), p. 291.

the former changes are always positive, whilst the dissipation may either increase or decrease after anneal. Hence no direct relation seems to exist between the influence of anneal upon the room temperature frequency and upon the dissipation at the same temperature.

The annealing temperatures at which the permanent increase of frequency begins to be observed show that this effect in some cases may be related to the recrystallisation of the specimen. On the other hand the changes in dissipation, which have also been observed in copper single crystals, may be associated with some changes in the mobility of the dislocations.

For platinum the values of Q_m^{-1} have been plotted in Fig. 19 in a logarithmic

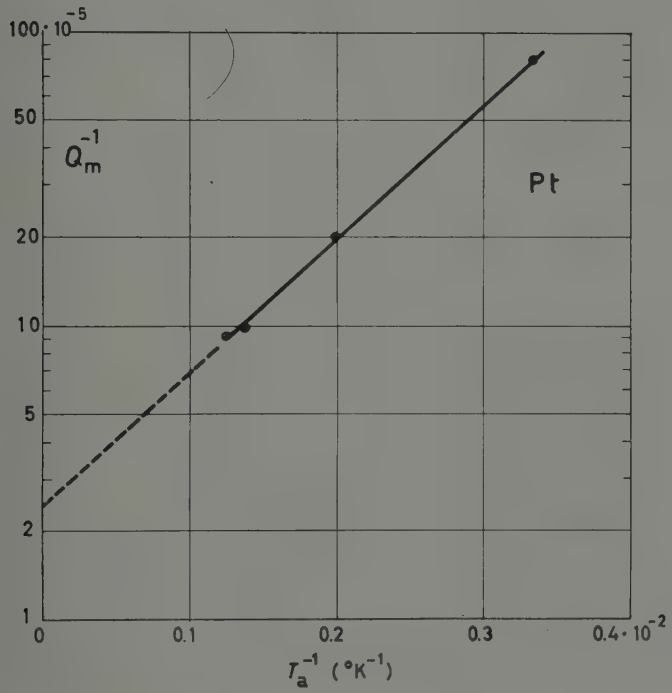

Fig. 19. – Exponential relation between the height of the dissipation peak and the inverse of the annealing temperature for platinum.

scale against the inverse of the annealing temperature T_a. As the experimental points fall on a straight line, the number of residual active dislocations is related to the annealing temperature by an Arrhenius equation, with an activation energy of about 0.09 eV. This energy is nearly one half of the activation energy associated with the relaxation effect (Table III). The reason why at room temperature dislocations move near their equilibrium position without annealing, is that the time required to form a kink is of the order

of $\overline{\tau}_0$, that is $\sim 10^{-12}$ s, and the probability of an energy fluctuation higher than $\overline{W}$ is large for such a short time. To make the dislocations inactive with respect to the relaxation effect, the energy fluctuations required must last a much longer time, and the corresponding probability is small at room temperature, notwithstanding the fact that the energy barrier seems to be lower than $\overline{W}$.

The influence of cold work on the relaxation effect (Fig. 11-14) is explained as easily as that of anneal, by an increase of the number of active dislocations. A temperature-independent *decrease* of vibration frequency after cold work is very evident in gold (Fig. 12, upper diagram) and in platinum (Fig. 14, upper diagram). This decrease is due to an effective change of the elastic coefficient, as the values of frequency given in Fig. 11-14, (upper diagrams, curves B) have been corrected for the thickness reduction of the plate and for the increase in diameter, due to the permanent strain.

For a given vibration frequency a slight dependence of the temperature T_m upon the annealing temperature T_a has already been observed in previous experimental work ([31]). As T_m decreases together with Q_m^{-1} when T_a increases, it was concluded that the dislocations more easily removed in copper are those associated with the largest values of the relaxation time, as the average value $\overline{\tau}$ is decreased by anneal. The same dependence is observed in the four metals, being very evident in silver (Fig. 6, lower diagram) and in platinum (Fig. 9, lower diagram). It may be added that cold-work has apparently the opposite effect, as T_m for a given frequency increases slightly with the amount of cold-work and therefore with the height of the dissipation peak (Fig. 11 and 14, lower diagram).

The most significant parameter to which the observed changes of T_m may be related is the total number of dislocations which are active in the relaxation effect. In the Fuoss-Kirkwood approximation this number is proportional to the ratio Q_m^{-1}/γ, which may be computed from the data of Tables II and IV. The diagrams of Fig. 20 show that the temperature of the peak increases regularly with the number of dislocations; in the narrow range of values considered this dependence may be satisfactorily approximated by a straight line. The changes of $\overline{\tau}$ corresponding to a given change of the total number of active dislocations may be easily computed from the diagrams of Fig. 20 and from the values of the activation energy given in Table III. For silver when Q_m^{-1}/γ goes from $1500 \cdot 10^{-5}$ to $750 \cdot 10^{-5}$ the value of $\overline{\tau}$ is decreased by a factor of about 5. In platinum, when Q_m^{-1}/γ goes from $500 \cdot 10^{-5}$ to $250 \cdot 10^{-5}$ the characteristic time $\overline{\tau}$ is reduced by a factor of about 2 ([32]).

([31]) See the papers quoted in footnotes ([2]), ([5]) and ([11]).

([32]) In the paper quoted in footnote ([2]), p. 312, a factor of 1.5 was computed for the change of $\overline{\tau}$ in copper. This value is correct, but one must read: « The change in the value of $(T_m^{-1} - T^{-1})$ is of the order of $0.3 \cdot 10^{-3}$ °K^{-1} ».

The above results show that anneal and cold work are effective in changing both the number of active dislocations and their average relaxation time. In fact when the number is reduced by a factor of 2, the characteristic time may be reduced by a larger factor.

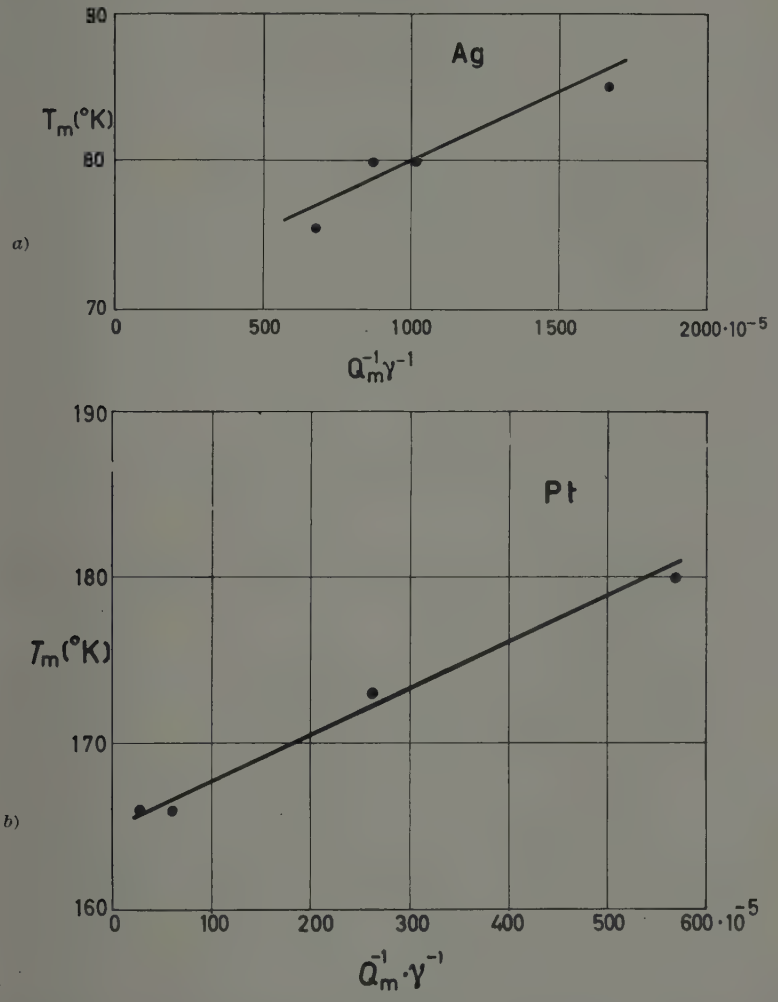

Fig. 20. – Dependence of the temperature of the peak upon the total number of active dislocations. Silver, experimental data reduced to the vibration frequency of 50 kHz; platinum, vibration frequency of 9.6 kHz.

The experimental evidence which is presently available is not sufficient to decide if the changes of $\bar{\tau}$, which are observed at low temperatures, are due to corresponding changes of the limiting value $\bar{\tau}_0$ of the characteristic time or, as it is more probable, to a slight variation of the activation energy. To give

a satisfactory answer to this question the activation energy must be measured after different anneals with an accuracy of the order of a few units percent.

3·5. *Subsidiary peaks.* – The dissipation curves for palladium (Fig. 3, 8 and 13, lower diagrams) and those for platinum (Fig. 4, 9 and 14, lower diagrams) clearly show a subsidiary peak below the temperature of the main maximum, of the same type as that found by NIBLETT and WILKS in high purity (99.999%) polycrystalline copper, and successively investigated in the same material by PARÈ and by others [33].

In silver and gold the existence of a N.W. peak is less obvious (Fig. 1, 2, 6, 7, 11, 12, lower diagrams). This is partly due to the fact that the experimental values are plotted against temperature. As it has been shown [34] the dissipation curves due to a symmetrical relaxation spectrum are not symmetrical with respect to T but to T^{-1}. When the values of the dissipation are plotted against the « *symmetrical* » variable $\overline{W}k^{-1}(T_m^{-1} - T^{-1})$, a flat « bump » is observed in the low temperature branch of the curves for silver (Fig. 15) and gold (Fig. 16).

The N.W. peak is clearly related to the dislocation motion, like the main one, as it is reduced or cancelled by anneal (Fig. 6-9, lower diagram) and increased or reintroduced by cold work (Fig. 11-14).

To separate the N.W. peak from the main maximum, the experimental values of dissipation have been plotted against T^{-1} as it is shown in Fig. 21 (heavy line). A Fuoss-Kirkwood's curve (eq. (9)) has been drawn to fit the experimental points, and the values of T_m, $\overline{W}$, $\overline{\tau}_0$ and γ given in the Tables II, III and IV have been employed to this purpose (Fig. 21, dotted line A). The difference between the experimental points of the low-temperature branch and the A curve has been computed, and two maxima have been found for $T^{-1} = 0.99 \cdot 10^{-2}\ ^\circ\mathrm{K}^{-1}$ and $T^{-1} = 1.37 \cdot 10^{-2}\ ^\circ\mathrm{K}^{-1}$. A second Fuoss-Kirkwood's curve has been drawn choosing the temperature of the first subsidiary maximum T'_m and the new value of the product $\gamma' \overline{W}' k^{-1}$ by a trial and error method, to fit the computed differences in the best possible way (Fig. 21, dotted line B). The procedure has been repeated for the second subsidiary maximum, and a third curve of the type (9) has been drawn through the new differences (Fig. 21, dotted line C). In this way the experimental dissipation curve has been approximated by the sum of three « *symmetrical* » curves, corresponding to symmetrical relaxation spectra.

The same procedure has been followed whenever possible for the other metals, and for the different vibration frequencies. The results obtained, that is the temperatures T'_m and T''_m of the two subsidiary peaks, their heights

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(33) See the papers quoted in footnote (11) and (13).

(34) See the papers quoted in footnote (2), p. 304 and in footnote (17).

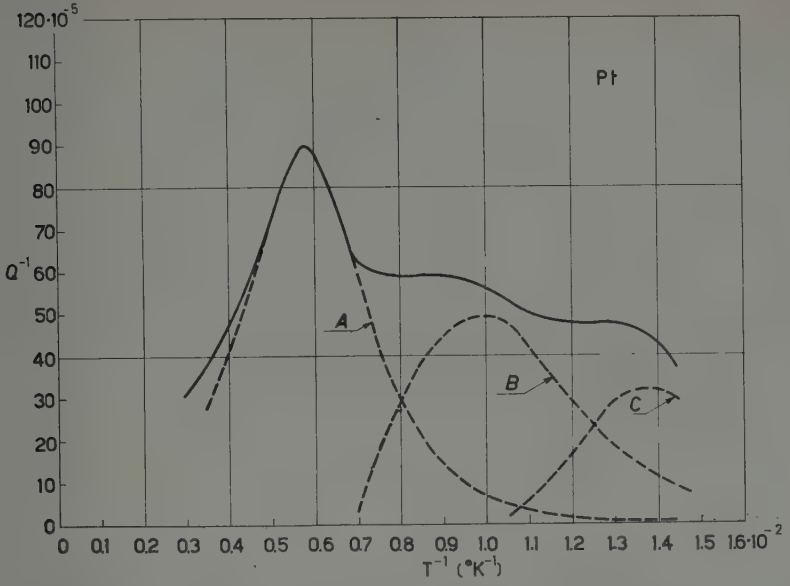

Fig. 21. – Separation of the N.W. peaks from the main dissipation maximum. Heavy line, experimental data; dotted lines *A*, *B*, *C*, computed Fuoss-Kirkwood's curves.

TABLE VI. – *Experimental data for the subsidiary peaks.*

Metal	Last treatment (*)	Frequency at T_m (kHz)	Main peak			First subsidiary peak			Second subsidiary peak		
			T_m (°K)	Q_m^{-1}	$\gamma\overline{W}k^{-1}$ (°K)	T'_m (°K)	$(Q'_m)^{-1}$	$\gamma'\overline{W}'k^{-1}$ (°K)	T''_m (°K)	$(Q''_m)^{-1}$	$\gamma''\overline{W}''k^{-1}$ (°K)
Silver (**)	(*b*)	421.4	91.0	$288\cdot10^{-5}$	430	61	$120\cdot10^{-5}$	460	—	—	—
Gold	(*a*)	15.99	117.7	202	635	<69	>90	—	—	—	—
	(*a*)	55.60	127.7	221	635	<74	>110	—	—	—	—
Palladium	(*a*)	24.92	180.0	30	650	107	20	650	<71	$>13\cdot10^{-5}$	—
	(*a*)	88.35	186.5	59	650	111.5	36	650	<80	>20	—
	(*a*)	387.9	200.0	75	650	—	>32	—	—	—	—
Platinum	(*a*)	9.60	173.2	90	780	101	49	630	73	32	630
	(*a*)	42.41	201.0	116	780	111	65	630	–	–	—

(*) For the details of thermal and mechanical treatments, see Table I.
(**) Measurements made on specimen no. 3.

$(Q'_m)^{-1}$ and $(Q''_m)^{-1}$ and the values of the product $\gamma' \overline{W}' k^{-1}$ for the first subsidiary peak are listed in Table VI.

The experimental data are too limited for a systematic evaluation of the activation energy $\overline{W}'$ associated with the first N.W. peak. However, the energy $\overline{W}' = 0.16$ eV which may be computed for palladium from the values of frequency and temperature given in Table VI, and the energy $\overline{W}' = 0.11$ eV which is obtained for platinum, are both consistent with the hypothesis that at high temperatures the corresponding values of the characteristic time are of the same order as the period of the Debye's frequencies. As it was found for copper, these energies are considerably smaller than the energies of the main peak. In silver and gold the N.W. peaks fall below the experimental range of temperatures and no information can be derived from Table VI on their activation energies.

4. – Conclusions.

The fundamental parameters $\overline{W}$ and $\overline{\tau}_0$ of the relaxation effect due to dislocations in silver, gold, palladium and platinum have values near to those found for the same parameters in other f.c.c. metals. This seems to point out that the motion of dislocations has an « *intrinsic* » character, essentially independent of the impurity content, in agreement with the present theoretical treatments of the same problem .

From a quantitative standpoint, the agreement betwen experiment and theory is satisfactory as far as the activation energy is concerned. The evaluation of the characteristic time must be improved, as the present values hardly give the right order of magnitude. In this respect the theory must also be extended to account for the wide spectrum of characteristic times which is always associated with the effect, and for the relation between the total number of dislocations and the value of $\overline{\tau}$ at low temperatures.

The asymmetry of the dissipation *vs.* temperature curves is a general property of the effect, and is possibly due to subsidiary peaks on the low temperature side of the main dissipation maximum. These peaks are of the same type as that found by NIBLETT and WILKS in copper. The dislocations responsible for this asymmetry seem to have a lower activation energy than those associated with the main peak.

The influence of anneal and cold work on the relaxation effect is now satisfactorily understood. The new results on silver, gold, palladium and platinum have removed an apparent contradiction with theory, showing that the temperature-independent increase of frequency which is produced by anneal, and the opposite effect due to cold work are not related to the change observed in dissipation but are probably due to recrystallization effects.

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Modèle semi-phénoménologique du noyau basé sur l'interaction nucléons-mésons π (*) (**).

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CONTENTS. — **1.** Introduction. – **2.** Elimination partielle des mésons. – **3.** Introduction d'un potentiel scalaire. – **4.** Autre méthode d'élimination des mésons. – **5.** Résolution formelle de l'eq. (3.1); construction de la fonction d'état générale. – **6.** Calcul de la source du champ V_α. – **7.** Calcul d'un exemple. – **8.** Niveaux d'énergie. – **9.** Parité. – **10.** Utilisation des fonctions d'onde; moments magnétiques. – APPENDICE *A*) Unités et symboles. – *B*) Champs d'opérateurs mésoniques $V_\alpha(\mathbf{x})$. – *C*) Parité M. *D*) Quelques propriétés utilisées des éléments de matrice de $V_\alpha(\mathbf{x})$.

1. – Introduction.

Pour autant que la théorie des champs soit aussi fructueuse dans le cas des nucléons que dans celui des électrons, les solutions stationnaires de l'équation

$$(1.1) \qquad i\frac{\partial \Psi}{\partial t} = \left\{ \int \mathrm{d}^3x (\nabla V_\alpha \cdot \nabla V^\alpha + \Pi^\alpha \Pi_\alpha + \mu^2 V_\alpha V^\alpha) + \right.$$
$$\left. + \int \mathrm{d}^3x \varphi^* \left(-\frac{1}{2m} \Delta + \frac{f}{\mu} \sigma \cdot \nabla V_\alpha \tau^\alpha \right) \varphi \right\} \Psi,$$

apporteraient la contribution essentielle à la description des noyaux. (1.1) est la première approximation non-relativiste communément admise de l'équation

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fournie par la théorie du couplage P.S.-P.V. et devrait suffire pour les basses énergies. Malheureusement, ses solutions ne sont pas encore connues, et l'on ignore même s'il en existe. Pourtant les mésons π jouent un rôle prépondérant dans la liaison des nucléons au sein des noyaux. Les expériences de diffusion pions-nucléons révèlent une forte interaction et parmi toutes les particules ayant une forte interaction avec les nucléons, les mésons π sont celles dont la masse est de loin la plus petite, ce qui entraîne qu'ils engendrent les forces de rayon d'action le plus grand. D'autre part, les bons résultats obtenus en introduisant un potentiel moyen dans les modèles phénoménologiques à particules indépendantes [1] deviennent incompréhensibles si l'on considère des forces de rayon d'action beaucoup plus petit que celui des forces π-mésoniques. Par là, seuls les mésons π restent en cause.

Un essai de dériver de l'interaction méson π-nucléon un modèle simple décrivant le comportement des noyaux sphériques moyens et lourds aussi bien que le modèle phénoménologique en couches, est par conséquent justifiable. En termes plus précis, il s'agit de trouver l'origine mésonique des deux termes essentiels du modèle en couches: le potentiel moyen et le couplage « spin-orbite »; c'est le sujet que nous abordons ici.

La méthode des forces statiques à deux corps, utilisée fructueusement pour le deutéron [2], conduit pour les noyaux lourds à un problème d'une extrême complexité [3], et nécessite l'introduction d'un cut-off; aussi l'éviterons-nous en utilisant une méthode simple qui conduit rapidement à un modèle en couches, contenant un terme jouant le même rôle que le couplage $\mathbf{L}\cdot\mathbf{S}$. Le potentiel scalaire ne vient pas directement, nous en verrons la raison à la Section **3**, et doit être introduit *ad hoc*.

Si l'on considère un système de A nucléons dans un état stationnaire, on peut transposer (1.1) dans l'espace de configuration et l'écrire directement:

$$
(1.2)\qquad
\begin{cases}
\boldsymbol{H}\Psi = (\boldsymbol{H}_M^0 + \boldsymbol{H}_N^0 + \boldsymbol{H}')\Psi = E\Psi\,, \\
\boldsymbol{H}_M^0 = \displaystyle\int \mathrm{d}^3x \big(\nabla \boldsymbol{V}_\alpha(\mathbf{x})\cdot\nabla \boldsymbol{V}^\alpha(\mathbf{x}) + \boldsymbol{\Pi}^\alpha(\mathbf{x})\boldsymbol{\Pi}_\alpha(\mathbf{x}) + \mu^2 \boldsymbol{V}_\alpha(\mathbf{x})\boldsymbol{V}^\alpha(\mathbf{x})\big)\,, \\
\boldsymbol{H}_N^0 = \displaystyle\sum_{a=1}^{A} -\frac{1}{2m}\Delta_a\,, \qquad \boldsymbol{H}' = \frac{f}{\mu}\sum_{a=1}^{A} \boldsymbol{\sigma}_a\cdot\nabla_a \boldsymbol{V}_\alpha(\mathbf{x}_a)\tau_a^\alpha\,, \\
\Psi = \Psi(\ldots;\, \mathbf{x}_a,\, s_a,\, t_a;\, \ldots;\, \ldots\, M_{klm}^\alpha \ldots)\,,
\end{cases}
$$

[1] K. Bleuler et Ch. Terreaux: *Helv. Phys. Acta*, **28**, fasc. 2-3 (1955); **30**, fasc. 2-3 (1957). M. Goeppert Mayer et J. H. D. Jensen: *Elementary Theory of Nuclear Shell Structure*.

[2] L. Hulthén et M. Sugawara: *The two-nucleon problem*, in *Encyclopedia of Physics*, vol. **39**.

[3] A. M. Feingold: *Phys. Rev.*, **101**, 258 (1956); T. Terasawa: *Progr. Theor. Phys.* **22**, 150 (1960) et **23**, 87 (1960).

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$\mathbf{x}_a$, s_a, t_a = variables d'espace, de spin et d'isospin du $a^{\text{ième}}$ nucléon; M^{α}_{klm} = nombres d'occupation des mésons (*).

On peut éliminer partiellement de (1.2) les variables mésoniques, sans passer par les forces à deux corps. Il suffit de postuler un Ψ de forme particulière, puis de prendre la moyenne de $\boldsymbol{H}$ relativement aux mésons. On obtient par ce procédé un système d'équations différentielles régissant les fonctions qui décrivent le mouvement des nucléons.

2. – Elimination partielle des mésons.

Considérons une fonction d'état dans laquelle les variables nucléoniques et mésoniques sont séparées: $\Psi = \varphi(X, \eta)\,|\,M\rangle$; $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ symbolise les variables d'espace des A nucléons et η leurs variables de spin et d'iso-spin; $|\,M\rangle$ est une combinaison linéaire de vecteurs à nombre fixe de mésons (**); les normes des trois fonctions valent $(\Psi, \Psi) = (\varphi, \varphi) = \langle M\,|\,M\rangle = 1$. Introduisons Ψ dans l'éq. (1.2) et multiplions scalairement chaque membre par $\langle M\,|$; on obtient l'équation différentielle:

$$\sum_{a=1}^{A}\left(-\frac{1}{2m}\Delta_a + \frac{f}{\mu}\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_a V_\alpha(\mathbf{x}_a)\tau_a^\alpha\right)\varphi(X,\eta) = E_N\varphi(X,\eta)\,, \tag{2.1}$$

où

$$V_\alpha(\mathbf{x}_a) = \langle M\,|\,\boldsymbol{V}_\alpha(\mathbf{x}_a)\,|\,M\rangle\,, \qquad E_N = E - \langle M\,|\,\boldsymbol{H}^0_M\,|\,M\rangle\,.$$

Les fonctions $V_\alpha(\mathbf{x}) = \langle M\,|\,\boldsymbol{V}_\alpha\,|\,M\rangle$ sont inconnues car on ne connaît pas $|\,M\rangle$, mais on peut établir une équation différentielle qu'elles doivent satisfaire. L'éq. (1.1) est une première approximation non-relativiste d'une équation hamiltonienne équivalente au système d'équations de mouvement:

$$\left(\gamma^\nu\partial_\nu + m + \frac{f}{\mu}\gamma^5\gamma^\nu\partial_\nu\boldsymbol{V}_\alpha\tau^\alpha\right)\boldsymbol{\Psi}(x) = 0\,, \tag{2.2}$$

$$(\Box - \mu^2)\boldsymbol{V}_\alpha(x) = \frac{f}{\mu}\,\partial_\nu(\overline{\boldsymbol{\Psi}}\gamma^5\gamma^\nu\tau_\alpha\boldsymbol{\Psi})\,. \tag{2.3}$$

En prenant la même approximation non-relativiste de (2.3) et transposant la source dans l'espace de configuration des nucléons:

$$(\Delta - \mu^2)\boldsymbol{V}_\alpha(\mathbf{x}) = -\frac{f}{\mu}\sum_{a=1}^{A}\delta(\mathbf{x}-\mathbf{x}_a)\tau_a^\alpha\sum_{i=1}^{3}\sigma_a^i\left(\frac{\delta}{\partial x^i} + \frac{\partial}{\partial x^i}\right) = \varrho_\lambda \quad (***)\,.$$

(*) Dans l'appendice sont rassemblées les notations particulières, des définitions, et quelques propriétés des opérateurs utilisés. Etant donné sa brièveté et son caractère de memento, nous l'indiquons en bloc dans les renvois.

(**) Appendice.

(***) $\delta/\partial x^i$ = dérivée à gauche et $\partial/\partial x^i$ = dérivée à droite.

La valeur moyenne de cette équation pour la fonction Ψ donne l'équation cherchée:

$$(\Psi, (\Delta - \mu^2) V_\alpha(\mathbf{x}) \Psi) = (\varphi, \varphi)(\Delta - \mu^2) \langle M | V_\alpha(\mathbf{x}) | M \rangle ,$$

$$(\Psi, \varrho_\alpha \Psi) = \langle M | M \rangle \sum_\eta \int_X \varphi^*(X, \eta) \varphi(X, \eta) ,$$

d'où

$$(2.4) \quad (\Delta - \mu^2) V_\alpha(\mathbf{x}) = -\frac{f}{\mu} \sum_{a=1}^{A} \nabla \cdot \sum_\eta \int_X \varphi^*(X, \eta) \tau_a^\alpha \boldsymbol{\sigma}_a \varphi(X, \eta) \delta(\mathbf{x} - \mathbf{x}_0) = \varrho_\alpha(\mathbf{x}) .$$

Les éq. (2.1) et (2.4) constituent un système d'équations différentielles couplées dont les solutions présentent une particularité: prenons en une, $\{\varphi(X, \eta), V_\alpha(\mathbf{x})\}$, correspondant à l'énergie E_N. La fonction $\tilde{\varphi}(X, \eta) = \varphi(-X, \eta)$ satisfait naturellement

$$(2.5) \quad \sum_{a=1}^{A} \left(-\frac{1}{2m} \Delta_a - \frac{f}{\mu} \boldsymbol{\sigma}_a \cdot \nabla_a V_\alpha(-\mathbf{x}_a) \tau_a^\alpha \right) \tilde{\varphi}(X, \eta) = E_N \tilde{\varphi}(X, \eta) ,$$

pour la même valeur E_N; la source engendrée par $\tilde{\varphi}$ est:

$$\tilde{\varrho}_\alpha(\mathbf{x}) = -\varrho_\alpha(-\mathbf{x})$$

si φ donne $\varrho_\alpha(\mathbf{x})$; $\tilde{\varrho}_\alpha$ est source d'un nouveau potentiel $\tilde{V}_\alpha(\mathbf{x}) = -V_\alpha(-\mathbf{x})$. En introduisant $\tilde{V}_\alpha$ dans (2.5) on retrouve la forme (2.1). Ainsi, le couple $\{\tilde{\varphi}(X, \eta), \tilde{V}_\alpha(\mathbf{x})\}$ est aussi solution du système (2.1-4) pour la valeur E_N de l'énergie; c'est une dégénérescence, d'un type particulier.

En supposant que $\Psi = \varphi | M \rangle$ représente un état d'un noyau, ou inversément que les états des noyaux sont représentables par des fonctions où les variables mésoniques et nucléoniques son séparées, le système d'équations (2.1-4) constituerait un modèle permettant de décrire le mouvement des nucléons. Les équations de ce modèle seraient exactement, d'autre part, celles que fournit la théorie quantique classique du couplage pseudo-vectoriel, qui consiste à envisager classiquement les équations de mouvement (2.2-3) des opérateurs. Selon ce point de vue, après une approximation non-relativiste et statique, (2.2-3) mènent à:

$$(2.6) \quad \left(-\frac{1}{2m} \Delta + \frac{f}{\mu} \boldsymbol{\sigma} \cdot \nabla V_\alpha(\mathbf{x}) \tau^\alpha \right) \chi_a(\mathbf{x}) = E_a \chi_a(\mathbf{x}) ,$$

$$(2.7) \quad (\Delta - \mu^2) V_\alpha(\mathbf{x}) = -\frac{f}{\mu} \sum_{a=1}^{A} \nabla \cdot (\chi_a^* \boldsymbol{\sigma}_a \tau_a^\alpha \chi_a(\mathbf{x})) = \varrho_\alpha ,$$

où a numérote les nucléons du système.

Les variables $\mathbf{x}_\alpha$ étant séparables directement dans (2.1), on trouve (2.6) en posant $\varphi(X, \eta) = \chi_1(\mathbf{x}_1) \dots \chi_A(\mathbf{x}_A)$; du même coup, ϱ_α de (2.4) prend la forme de celui de (2.7) si on norme les χ_α.

Un tel modèle, bien sûr, n'a pas de sens physique car les fonctions d'onde des nucléons n'ont pas une parité définie, et par conséquent φ et Ψ non plus; ce fait provient de ce que $[\pi, \boldsymbol{\sigma} \cdot \nabla V_\alpha(\mathbf{x})] \neq 0$ (π étant l'opérateur de parité défini par $\pi \chi(\mathbf{x}) = \chi(-\mathbf{x})$) (*), et il est lié au caractère pseudo-scalaire du champ des mésons π. On serait tenté d'utiliser la dégénérescence vue avant, et de la lever en introduisant un terme supplémentaire après diagonalisation des solutions selon π. En fait c'est impossible car on a affaire à une dégénérescence des solutions du système (2.1-4), et non pas de (2.1) seule, et on ne peut combiner linéairement deux solutions $\{\varphi, V_\alpha\}$ et $\{\tilde{\varphi}, \tilde{V}_\alpha\}$ car φ intervient non linéairement dans la source ϱ_α. A titre illustratif on pourrait dire que les deux solutions représentent deux « noyaux » fictifs, « miroirs » l'un de l'autre, et seul un mélange des deux peut correspondre à un noyau réel.

Ce mélange, impossible classiquement, peut être essayé en conservant les mésons dans la fonction d'état. Remarquons tout d'abord que la seconde solution $\{\tilde{\varphi}, \tilde{V}_\alpha\}$ peut être obtenue par le même procédé que la première, en introduisant dans (1.2) une fonction $\tilde{\Psi} = \tilde{\varphi} | \tilde{M} \rangle$ puis en prenant la moyenne relativement à $\langle \tilde{M} |$; si $| \tilde{M} \rangle$ est tel que $\langle \tilde{M} | V_\alpha(\mathbf{x}) | \tilde{M} \rangle = \langle M | -V_\alpha(-\mathbf{x}) | M \rangle$, alors $\tilde{\varphi}$ satisfait (2.5) avec le même spectre de valeurs propres que celui de (2.1). Le vecteur mésonique $| \tilde{M} \rangle$ peut être facilement défini si l'on fait une supposition, nécessaire par ailleurs. Nous postulons que les vecteurs $| M \rangle$ ne contiennent que des mésons S (moment angulaire $l = 0$ (*)). Les équations devant s'appliquer aux noyaux, il faut qu'elles aient la même symétrie spatiale qu'eux, c'est-à-dire la symétrie sphérique (nous ne considérons pas, ici, les noyaux déformés). Cela impose que les potentiels pseudo-scalaires ne dépendent que de la coordonnée $r = |\mathbf{x}|$: $\langle M | V_\alpha | M \rangle = V_\alpha(r)$, donc que $| M \rangle$ ne contienne que des mésons S (*). Cette hypothèse est compatible avec l'éq. (2.4) lorsque les niveaux J sont complets ($2J+1$ particules), car dans ce cas $\varrho_\alpha = \varrho_\alpha(r)$ (**); on peut, sans perdre en généralité, supposer que tous les niveaux sont pleins, car les écarts sont faibles dans un modèle en couches. Décomposons maintenant $| M \rangle$ en parties paire et impaire:

$$| M \rangle = \frac{1}{\sqrt{2}} (| + \rangle + | - \rangle),$$

$\boldsymbol{\Pi}_M | + \rangle = | + \rangle$, $\boldsymbol{\Pi}_M | - \rangle = - | - \rangle$, $\boldsymbol{\Pi}_M =$ opérateur de parité dans l'espace des mésons; $| + \rangle$ est une somme de vecteurs contenant un nombre pair de mé-

(*) Appendice.
(**) Voir Sect. 6.

sons S, et $|-\rangle$ un nombre impair. Substituons cette expressions dans la définition de V_α:

$$V_\alpha(r) = \langle M|\boldsymbol{V}_\alpha(\mathbf{x})|M\rangle = \tfrac{1}{2}\{\langle -|\boldsymbol{V}_\alpha|+\rangle + \langle +|\boldsymbol{V}_\alpha|-\rangle\} \quad (^*)\,.$$

Il est alors clair que:

$$\tilde{V}_\alpha(r) = \langle \tilde{M}|\boldsymbol{V}_\alpha(\mathbf{x})|\tilde{M}\rangle = -V_\alpha(|-\mathbf{x}|) = -V_\alpha(r)$$

si

$$|\tilde{M}\rangle = \boldsymbol{\Pi}_M|M\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\,.$$

Ainsi: $\tilde{\Psi} = \boldsymbol{\Pi}\Psi$, si $\boldsymbol{\Pi}$ est l'opérateur de parité dans l'espace de Hilbert nucléons-mésons (*). Les fonctions Ψ et $\tilde{\Psi}$ conduisent à des « modèles » différents ayant même spectre d'énergie, mais sans sens physique, pas plus que les fonctions elles-mêmes qui n'ont pas de parité. Malgré cette difficulté, il n'en reste pas moins que la séparation des variables n'est pas dénuée de sens puisqu'elle correspond à l'hypothèse que les noyaux sont approximativement représentables par un modèle en couches, ce qui est vérifié [4]. Aussi tentons nous l'essai suivant: Prenons Ψ et $\tilde{\Psi}$ comme fonctions de base pour un calcul de variation, et posons $\Phi(\alpha,\beta) = \alpha\Psi + \beta\tilde{\Psi}$ $(\alpha^*\alpha + \beta^*\beta = 1)$; $\Phi(\alpha,\beta)$ correspond à un mélange des « modèles miroirs ». Puis, calculons les valeurs de α et β qui rendent minimum la valeur moyenne $\langle\Phi|\boldsymbol{H}|\Phi\rangle$ de l'hamiltonien $\boldsymbol{H}$:

$$E(\alpha,\beta) = \langle\Phi|\boldsymbol{H}|\Phi\rangle = \alpha^*\alpha\langle\Psi|\boldsymbol{H}|\Psi\rangle + \alpha^*\beta\langle\Psi|\boldsymbol{H}|\tilde{\Psi}\rangle + \alpha\beta^*\langle\tilde{\Psi}|\boldsymbol{H}|\Psi\rangle + \beta^*\beta\langle\tilde{\Psi}|\boldsymbol{H}|\tilde{\Psi}\rangle\,.$$

On peut décomposer Ψ et $\tilde{\Psi}$ en toute généralité:

$$\Psi = \varphi(X,\eta)\cdot|M\rangle = (\varphi^+(X,\eta) + \varphi^-(X,\eta))\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = \frac{1}{\sqrt{2}}(\Psi^+ + \Psi^-)\,,$$

$$\tilde{\Psi} = \tilde{\varphi}(X,\eta)\cdot|M\rangle = \boldsymbol{\Pi}\tilde{\Psi} = \frac{1}{\sqrt{2}}(\Psi^+ - \Psi^-)\,,$$

où

$$\varphi^\pm(X,\eta) = \tfrac{1}{2}(\varphi(X,\eta) \pm \varphi(-X,\eta))\,, \qquad \Psi^\pm = \varphi^+|\pm\rangle + \varphi^-|\mp\rangle\,.$$

Posons $E^\pm = \langle\Psi^\pm|\boldsymbol{H}|\Psi^\pm\rangle$; comme $\langle\Psi^+|\boldsymbol{H}|\Psi^-\rangle = 0$ (**), on a

$$E(\alpha,\beta) = \tfrac{1}{2}(\alpha^*\alpha + \beta^*\beta)(E^+ + E^-) + \tfrac{1}{2}(\alpha^*\beta + \alpha\beta^*)(E^+ - E^-)\,.$$

(*) Appendice.

(4) K. Bleuler et Ch. Terreaux: *Helv. Phys. Acta*, **30**, 183 (1957).

(**) Car $[\boldsymbol{H}, \boldsymbol{\Pi}] = 0$.

Soient $E_0 = \frac{1}{2}(E^+ + E^-)$, $\Delta E = E^- - E^+$, $\alpha = |\alpha| \exp[i\omega]$, $\beta = |\beta| \exp[i\varkappa]$. La propriété $\alpha^*\alpha + \beta^*\beta = 1$ entraîne finalement:

$$E(\alpha, \beta) = E_0 - |\alpha| \sqrt{1 - |\alpha|^2} \cos(\omega - \varkappa) \cdot \Delta E .$$

$E(\alpha, \beta)$ est extrémale pour $|\alpha| = 1/\sqrt{2}$ et $\varkappa = \omega$, $\omega + \pi$. Comme Φ est définie à une phase près, on peut poser $\omega = 0$. D'où:

$$E(\alpha, \beta) \text{ est minimum pour } \begin{cases} \alpha = \quad \beta = \dfrac{1}{\sqrt{2}} & \text{si} \quad \Delta E > 0 , \\ \alpha = -\beta = \dfrac{1}{\sqrt{2}} & \text{si} \quad \Delta E < 0 , \end{cases}$$

et

$$\Psi^+ = \Phi\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} (\Psi + \tilde{\Psi}) = \varphi^+(X, \eta) \cdot |+\rangle + \varphi^-(X, \eta) \cdot |-\rangle ,$$

$$\Psi^- = \Phi\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} (\Psi - \tilde{\Psi}) = \varphi^+(X, \eta) \cdot |-\rangle + \varphi^-(X, \eta) \cdot |+\rangle .$$

Ainsi, les fonction $\Phi(\alpha, \beta)$ rendant minimum l'énergie ont une parité déterminée; elles ont un sens physique et pourront être utilisées pour représenter des états de noyaux. La parité de l'état fondamental est déterminée par le signe de ΔE:

$$\text{Etat fondamental } \Psi^+ \text{ si } \Delta E > 0 ,$$

$$\text{Etat fondamental } \Psi^- \text{ si } \Delta E < 0 .$$

Cette question est traîtée à la Section **9**.

En appliquant $[\boldsymbol{H}^0_M, \boldsymbol{\Pi}] = [\boldsymbol{H}^0_N, \boldsymbol{\Pi}] = [\boldsymbol{H}', \boldsymbol{\Pi}] = 0$, ce qui entraîne

$$(\varphi^+, \boldsymbol{H}^0_N \varphi^-) = \langle + | \boldsymbol{H}^0_M | - \rangle = (\varphi^\pm, \langle + | \boldsymbol{H}' | - \rangle \varphi^\pm) = 0 ,$$

et en utilisant $(\varphi^+, \varphi^+) + (\varphi^-, \varphi^-) = \langle \pm | \pm \rangle = 1$, on obtient

$$\Delta E = \hbar\{(\varphi^+, \varphi^+) - (\varphi^-, \varphi^-)\} + 2i\{(\varphi^-, H'_I \varphi^+) - (\varphi^+, H'_I \varphi^-)\} , \tag{2.8}$$

où

$$\hbar = \langle - | \boldsymbol{H}^0_M | - \rangle - \langle + | \boldsymbol{H}^0_M | + \rangle , \qquad H'_I = \frac{f}{\mu} \sum_{a=1}^{A} \boldsymbol{\sigma}_a \cdot \boldsymbol{\nabla}_a V^I_\varkappa \tau^\alpha_a .$$

Le potentiel $V_\varkappa(r) = \langle M | \boldsymbol{V}_\alpha | M \rangle$ est évidemment réel, mais pas forcément, $\langle + | \boldsymbol{V}_\alpha | - \rangle$, qui n'est pas un élément de matrice diagonal; il reste $V^I_\varkappa =$ partie imaginaire de $\langle + | \boldsymbol{V}_\alpha | - \rangle$ dans ΔE.

Il est possible, par un raisonnement indépendant du précédent, de prévoir la forme des fonctions Ψ les plus simples qu'il convient d'utiliser. En introduisant ces fonctions dans (1.2) et en prenant des moyennes relativement aux mésons, on tombe sur des systèmes d'équations différentielles ressemblant à (2.1) mais dans lesquels on ne peut séparer les variables $\mathbf{x}_a$. Il est alors pratique d'utiliser les solutions de (2.1) qui sont une approximation des solutions de ces systèmes. Ce sujet est abordé à la Section 4.

3. – Introduction d'un potentiel scalaire.

Un premier test révèle qu'un modèle de noyau basé sur les éq. (2.1-4) ne donne pas des résultats satisfaisants. Pour avoir une idée du spectre d'énergie nous avons résolu le problème à une particule (2.6) en postulant différents $V_\alpha(r)$. Il s'avère qu'en aucun cas on obtient un nombre correct de niveaux et une bonne succession; il manque visiblement dans l'hamiltonien un terme scalaire $U(r)$.

Un tel terme apparaît pourtant dans les théories mésoniques. Si l'on aborde le problème du couplage entre nucléons en résolvant (1.2) au moyen de l'approximation des forces statiques à deux corps et d'un calcul de perturbation, on obtient un hamiltonien pour les nucléons seuls composé des termes d'ordre 0, 2, 4, etc., du calcul de perturbation (les ordres impairs sont nuls) [2]. Les deux premiers termes peuvent également être obtenus par la théorie des champs classiques (non-quantifiés), mais pas ceux d'ordre supérieur. Le 4^{me} ordre contient un terme scalaire d'interaction à deux corps, et quelques autres dont l'apport à l'énergie totale est pratiquement nul si on prend la moyenne pour toutes les particules. Dans son travail sur le deutéron, L. Hulthén [2] montre qu'il faut tenir compte de ce terme scalaire pour obtenir des résultats satisfaisants. Il n'est pas étonnant que nous ayons besoin de ce potentiel $U(r)$; on peut montrer que l'équation hamiltonienne pour A nucléons ne contenant que les termes d'ordre 0 et 2 correspond à l'éq. (2.6) pour un nucléon plus une équation pour V_α. En outre, par la même opération, on peut évaluer que le terme scalaire du 4^{me} ordre (avec cut-off) donne un potentiel scalaire moyen d'ordre de grandeur semblable à celui des potentiels nucléaires mesurés [5].

Il se pourrait, soit dit à titre de remarque, qu'un perfectionnement de notre procédé donne le terme désiré, par exemple en introduisant dans (1.2) une fonction d'essai Ψ de plus de deux termes. Nous ne nous hasarderons pas dans une telle recherche; étant donné, d'une part, les considérations ci-dessus, d'autre part la nécessité où nous nous trouvons, nous introduisons *a priori*

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[5] P. Huguenin: communication privée.

un potentiel $U(r)$ dans l'éq. (2.6); avec cette adjonction phénoménologique notre modèle pseudo-scalaire est, par définition, basé sur le système d'équations

$$\left(H_0 + \frac{f}{\mu}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}V_\alpha\tau^\alpha\right)\chi_a(\mathbf{x}) = E_a\chi_a(\mathbf{x})\,, \tag{3.1}$$

$$(\Delta - \mu^2)V_\alpha(\mathbf{x}) = -\frac{f}{\mu}\sum_{a=1}^{A}\boldsymbol{\nabla}(\chi_a^*(\mathbf{x})\boldsymbol{\sigma}\tau^\alpha\chi_a(\mathbf{x}))\,, \tag{3.2}$$

où $H_0 = -(1/2m)\Delta + U(r)$, $U(r)$ à choisir en fonction des résultats expérimentaux.

Les fonctions d'état d'un noyau sont construites à partir des solutions de (3.1):

$$\Psi^\pm = \varphi^\pm(X,\eta)\,|+\rangle + \varphi^\mp(X,\eta)\,|-\rangle\,,$$

où

$$\varphi^\pm(X,\eta) = \tfrac{1}{2}(\varphi(X,\eta) \pm \varphi(-X,\eta))\,, \qquad \varphi(X,\eta) = \chi_1(\mathbf{x}_1)\ldots\chi_A(\mathbf{x}_A)\,.$$

Les fonctions $\varphi^+(X,\eta)$ et $\varphi^-(X,\eta)$ servent à décrire le mouvement des nucléons, mais elles ne sont pas des fonctions d'onde au sens courant du terme. Nous voyons à la Section **10** comment les utiliser. Le fait qu'on ne puisse représenter un état de noyau par une fonction dont les mésons sont absents provient du caractère pseudo-scalaire du champ mésonique.

Remarque 1. Le potentiel $U(r)$ ne change en rien la structure du modèle, dérivée par le procédé de la Section **2**, qui a l'avantage de permettre l'utilisation du terme $\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\,V_\alpha\tau^\alpha$ dans un modèle à particules indépendantes.

Remarque 2. $U(r)$ englobe des effets qui ne semblent guère explicables au moyen des théories mésoniques actuelles, comme par exemple le « cœur répulsif ».

Remarque 3. On peut remplacer (3.1) dans la définition par

$$\left(H_0 + \frac{f}{\mu}\sum_{a=1}^{A}\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_a V_\alpha(\mathbf{x}_a)\tau_a^\alpha\right)\varphi(X,\eta) = E\varphi(X,\eta)\,, \tag{3.1'}$$

où

$$H_0 = \sum_{a=1}^{A}\left(-\frac{1}{2m}\,\Delta_a + U(r_a)\right).$$

4. – Autre méthode d'élimination des mésons.

Au lieu d'essayer au départ une solution approximative de (1.2) revêtant la forme la plus simple, soit le produit $\varphi(X,\eta)\,|M\rangle$, on peut partir du point de vue opposé, c'est-à-dire de la solution exacte formellement développée en

série:

$$\Psi = \sum_n \varphi_n(X, \eta)\,|n\rangle\,.$$

Les fonctions $\varphi_n(X, \eta)$ dépendent des variables nucléoniques, et les vecteurs $|n\rangle$ des variables mésoniques. Nous choisissons, pour les mésons, la représentation des nombres d'occupation en coordinnées sphériques. Les vecteurs $|n\rangle$ sont des combinaisons linéaires de vecteurs $|\ldots M^{\alpha}_{klm}\rangle$, M^{α}_{klm} = nombre d'occupation (*).

La solution exacte Ψ est donnée par un développement infini; l'idée de l'approximation consiste à ne retenir qu'un nombre fini de termes, mais sans choisir d'avance les fonctions de base φ_n, $|n\rangle$; à chaque nombre de termes retenus correspond un modèle, dans le cadre duquel on calcule ces fonctions. Nous choisirons un nombre de termes tel que le modèle obtenu soit le plus près possible d'un modèle en couches, l'existence de couches dans la structure des noyaux étant un fait acquit (1).

Le point de départ utilisé à la Section 2, soit la séparation des variables, équivaut à dire, dans la présente formulation, qu'on ne retient qu'un terme de la série. On est alors conduit à un modèle en couches pur, mais qui ne permet pas de concilier deux exigences physiquement nécessaires: parité déterminée des états et sphéricité. Le modèle le plus voisin permettant de satisfaire ces deux conditions est obtenu en gardant deux termes:

$$\Psi = \varphi_1(X, \eta)\cdot|1\rangle + \varphi_2(X, \eta)\cdot|2\rangle\,.$$

Les fonctions φ_n, $|n\rangle_{n=1,2}$ ne sont pas absolument arbitraires: la relation $\mathbf{\Pi}\Psi = \mathbf{\Pi}_N\varphi_1\cdot\mathbf{\Pi}_M|1\rangle + \mathbf{\Pi}_N\varphi_2\cdot\mathbf{\Pi}_M|2\rangle = \pm\,\Psi$ n'est satisfaite que si $\mathbf{\Pi}_N\varphi_n = \pm\,\varphi_n$, $\mathbf{\Pi}_M|n\rangle = \pm\,|n\rangle$; de plus, pour que les équations aient la symétrie sphérique, il faut que les éléments de matrice $\langle n|\boldsymbol{V}_{\alpha}(\mathbf{x})|n'\rangle$ soient des fonctions de $r = |\mathbf{x}|$ exclusivement; nous exigeons, en conséquence (*), que $|n\rangle$ = combinaison linéaire de vecteurs $|\ldots M^{\alpha}_{klm}\ldots\rangle$ où $M^{\alpha}_{klm} = 0$ si $l \neq 0$ (on ne tient compte que des mésons S).

En remplaçant les indices 1 et 2 par (+) et (—), qui indiquent la parité de la fonction, les deux vecteurs d'état approximatifs s'écrivent:

$$\Psi^{+} = \varphi^{+}(X, \eta)\,|+\rangle + \varphi^{-}(X, \eta)\,|-\rangle\,, \tag{4.1}$$

$$\Psi^{-} = \varphi^{+}(X, \eta)\,|-\rangle + \varphi^{-}(X, \eta)\,|+\rangle\,. \tag{4.2}$$

Introduisons (4.1) dans l'éq. (1.2), et multiplions scalairement chaque

(*) Appendice.

membre par $\langle + |$, puis par $\langle - |$; il vient:

$$\left(\sum_{a=1}^{A} -\frac{1}{2m}\Delta_a\right)\varphi^+(X,\eta) + \left(\frac{f}{\mu}\sum_{a=1}^{A}\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_a\frac{\langle + |\boldsymbol{V}_\alpha(\mathbf{x}_a)|-\rangle}{\langle + |+\rangle}\tau_a^\alpha\right)\varphi^-(X,\eta) =$$

$$= \left(E - \frac{\langle + |\boldsymbol{H}_M^0|+\rangle}{\langle + |+\rangle}\right)\varphi^+(X,\eta)\,,$$

$$\left(\sum_{a=1}^{A} -\frac{1}{2m}\Delta_a\right)\varphi^-(X,\eta) + \left(\frac{f}{\mu}\sum_{a=1}^{A}\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_a\frac{\langle - |\boldsymbol{V}_\alpha|(\mathbf{x}_a)|+\rangle}{\langle - |-\rangle}\tau_a^\alpha\right)\varphi^+(X,\eta) =$$

$$= \left(E - \frac{\langle - |\boldsymbol{H}_M^0|-\rangle}{\langle - |-\rangle}\right)\varphi^-(X,\eta)\,,$$

car $\langle \pm |\boldsymbol{V}_\alpha| \pm\rangle = 0$ et $\langle + |\boldsymbol{H}_M^0|-\rangle = 0$ (*).

On peut éliminer les valeurs $\langle + |+\rangle$ et $\langle - |-\rangle$ des équations au moyen de la transformation:

$$\tilde{\varphi}^\pm = \sqrt{\langle \pm |\pm\rangle}\,\varphi^\pm\,, \qquad |\tilde{\pm}\rangle = \frac{1}{\sqrt{\langle \pm |\pm\rangle}}|\pm\rangle\,,$$

qui laisse la norme de Ψ invariante. Cette liberté est naturelle, car les normes individuelles de $\varphi^\pm$ et $|\pm\rangle$ n'interviennent jamais isolément, et seules les normes des produits $\varphi|\rangle$ ont un sens physique de probabilité.

En effectuant cette transformation et posant

$$H_0 = \sum_{a=1}^{A} -\frac{1}{2m}\Delta_a\,, \qquad E_M^\pm = \langle \pm |\boldsymbol{H}_M^0|\pm\rangle\,, \qquad V_\alpha(r) = \langle + |\boldsymbol{V}_\alpha|(\mathbf{x})|-\rangle\,,$$

on a

$$(4.3)\qquad \begin{cases} H_0\varphi^+(X,\eta) + \left(\dfrac{f}{\mu}\displaystyle\sum_{a=1}^{A}\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_a V_\alpha(r_a)\tau_a^\alpha\right)\varphi^-(X,\eta) = (E - E_M^+)\varphi^-(X,\eta)\,, \\ H_0\varphi^-(X,\eta) + \left(\dfrac{f}{\mu}\displaystyle\sum_{a=1}^{A}\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla} V_\alpha^*(r_a)\tau_a^\alpha\right)\varphi^+(X,\eta) = (E - E_M^-)\varphi^-(X,\eta)\,. \end{cases}$$

En introduisant (4.2) dans (1.2), on obtient un système d'équations analogue pour les états de parité (—):

$$(4.4)\qquad \begin{cases} H_0\varphi^-(X,\eta) + \left(\dfrac{f}{\mu}\displaystyle\sum_{a=1}^{A}\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_a V_\alpha^*(r_a)\tau_a^\alpha\right)\varphi^-(X,\eta) = (E - E_M^-)\varphi^+(X,\eta)\,, \\ H_0\varphi^-(X,\eta) + \left(\dfrac{f}{\mu}\displaystyle\sum_{a=1}^{A}\boldsymbol{\sigma}_a\cdot\boldsymbol{\nabla}_a V_\alpha(r_a)\tau_a^\alpha\right)\varphi^+(X,\eta) = (E - E_M^+)\varphi^-(X,\eta)\,. \end{cases}$$

(*) Appendice.

Il est essentiel que les systèmes soient différents (on passe d'un à l'autre en intervertissant E_M^+ et E_M^-, ainsi que V_α et V_α^*). S'ils étaient identiques, il y aurait dégénérescence par rapport à la parité, car on pourrait construire une solution Ψ^+ et une autre Ψ^- à partir des mêmes fonctions φ^+ et φ^-, et des mêmes kets $|+\rangle$ et $|-\rangle$.

Les systèmes d'éq. (4.3) et (4.4) sont très semblables à l'éq. (2.1); il y a tout lieu de croire que leurs spectres d'énergie ne seront guère différents, et que le même manque de potentiel scalaire se fera sentir. En conséquence nous introduisons ici, également, un potentiel phénoménologique $U(r)$; il n'y a pas d'équivoque quant à sa place, étant donné ses caractères réels et scalaires; il suffit de l'ajouter au laplacien, et redéfinir

$$H_0 = \sum_{a=1}^{A} \left(-\frac{1}{2m} \Delta_a + U(r_a) \right). \tag{4.5}$$

Les systèmes (4.3) et (4.4) présentent l'inconvénient qu'on ne peut les ramener à une équation pour une particule en séparant les variables; si on pose $\varphi(X, \eta) = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$, fonction à deux composantes, (4.3), par exemple, s'écrit

$$\begin{pmatrix} H_0 & H'_R \\ H'_R & H_0 \end{pmatrix} \varphi(X, \eta) + \begin{pmatrix} -h/2 & iH'_I \\ -iH'_I & h/2 \end{pmatrix} \varphi(X, \eta) = E_N \varphi(X, \eta) ,$$

où

$$H'_{\substack{R \\ I}} = \frac{f}{\mu} \sum_{a=1}^{A} \boldsymbol{\sigma}_a \cdot \boldsymbol{\nabla}_a V_\alpha^{\substack{R \\ I}}(r_a) \tau_a^\alpha , \qquad h = E_M^- - E_M^+ ,$$

$$V_\alpha^R = \operatorname{Re} V_\alpha(r) , \qquad V_\alpha^I = \operatorname{Im} V_\alpha(r) , \qquad E_N = E - \tfrac{1}{2}(E_M^+ + E_N^-) .$$

L'opérateur $\begin{pmatrix} -h/2 & iH'_I \\ -iH'_I & h/2 \end{pmatrix}$ ne peut se mettre sous forme d'une somme d'opérateurs partiels relatifs à un nucléon, d'une part, et d'autre part empêche de sommer les deux équations d'un système.

Il faut faire une approximation pour ramener le tout à un modèle à particules indépendantes. Le terme gênant est probablement petit devant l'autre, qui contient le $U(r)$ assurant l'essentiel de la liaison et la partie réelle de V_α. Négligeons ce terme pour le calcul des fonctions φ, et tenons-en compte ensuite par un calcul de perturbation du premier ordre. Les systèmes (4.3) et (4.4) deviennent identiques et on obtient, en sommant les équations

$$(H_0 + H'_R)\varphi_0(X, \eta) = E_N^0 \varphi_0(X, \eta) , \tag{4.6}$$

où $\varphi_0 = \varphi_0^+ + \varphi_0^-$.

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On retrouve en (4.6) l'éq. (3.1′); donc la fonction φ utilisée à la Section 2 est une première approximation des solutions de (4.3) et (4.4), identique à φ_0.

En ce qui concerne l'énergie, le calcul de perturbation donne:

$$\Delta E^+ = \frac{h}{2}\{(\varphi_0^-, \varphi_0^-) - (\varphi_0^+, \varphi_0^+)\} + i\{(\varphi_0^+, H'_I\varphi_0^-) - (\varphi_0^-, H'_I\varphi_0^+)\},$$

$$\Delta E^- = -\Delta E^+ .$$

$\Delta E = E_N^- - E_N^+ = \Delta E^- - \Delta E^+ =$ écart entre niveaux à parité opposée. Ce ΔE est indentique à celui qui découle du calcul de variation (voir (2.8)).

5. – Résolution formelle de l'eq. (3.1); construction de la fonction d'état générale.

Nous résolvons l'éq. (3.1) au moyen d'un calcul de perturbation:

$$H\chi(\mathbf{x}) = (H_0 + H')\chi(\mathbf{x}) = E\chi(\mathbf{x}), \tag{3.1}$$

$H_0 = -(1/2m)\Delta + U(r)$ est l'hamiltonien sans perturbation, et $H' = (f/\mu)\boldsymbol{\sigma}\cdot\nabla V_\alpha(r)\tau^\alpha$ est le terme de perturbation.

Nous supposons en cela $\langle H'\rangle \ll \langle H_0\rangle$, ce qui se confirme dans la suite. Il est commode d'établir d'abord la forme des solutions. L'hamiltonien H commute avec les opérateurs de rotation de l'espace autour de l'origine, car Δ, $U(r)$ et $\boldsymbol{\sigma}\cdot\nabla V_\alpha(r)$ ont la symétrie sphérique; par conséquent $[H, \mathbf{J}^2]=[H, J_z]=0$, ce qui permet aux solutions χ d'être fonctions propres des opérateurs de moment cinétique total $\mathbf{J}^2$ et J_z. Par contre, H ne commute pas avec l'opérateur π de réflexion de l'expace (parité): $[H_0, \pi] = 0$, car $[U(r), \pi] = [\Delta, \pi] = 0$, mais $[H', \pi] \neq 0$. Posons

$$\sigma = \boldsymbol{\sigma}\cdot\frac{\mathbf{x}}{r}, \qquad \boldsymbol{\sigma}\cdot\nabla V_\alpha(r) = \sigma\frac{\mathrm{d}V_\alpha}{\mathrm{d}r} = \sigma V'_\alpha(r) .$$

La propriété:

$$\sigma\pi + \pi\sigma = 0 \tag{5.1}$$

entraîne que $\sigma V'_\alpha(r)$, donc H', ne commutent pas avec π. Les solutions χ n'ont par conséquent pas de parité déterminée. L'ensemble des fonctions des deux variables θ et φ (*) qui sont fonctions propres de $\mathbf{J}^2$ et J_z, pour un couple de valeurs propres déterminées $\{J, M\}$, constituent un sous-espace vectoriel à

(*) $\mathbf{x} = (r, \theta, \varphi)$.

deux dimensions; puisque $[\mathbf{J}^2, \pi] = [J_z, \pi] = 0$, ce sous-espace est sous-tendu par les deux fonctions propres de π: $\Phi^+_{JM}(\theta, \varphi)$ et $\Phi^-_{JM}(\theta, \varphi)$; on peut ainsi écrire directement les solutions sous forme de combinaisons linéaires de ces fonctions au moyen de fonctions radiales:

$$\chi_{JM}(\mathbf{x}) = \frac{1}{r} f^+_J(r) \cdot \Phi^+_{JM}(\theta, \varphi) + \frac{1}{r} f^-_J(r) \cdot \Phi^-_{JM}(\theta, \varphi) . \tag{5.2}$$

Les fonctions angulaires $\Phi^\pm$ on deux composantes de spin, et les fonctions radiales $f^\pm$ en ont deux d'iso-spin.

On peut remplacer les indices $+$ et $-$ par le nombre quantique l, car $\Phi^\pm_{JM}$ sont fonctions propres de $\mathbf{L}^2$; nous notons dorénavant les fonctions angulaires: $\Phi_{JMl=J-\frac{1}{2}\varepsilon}(\theta, \varphi)$ $(\varepsilon = \pm 1)$, ou encore $\Phi_{lMJ=l+\frac{1}{2}\varepsilon}$ $(\varepsilon = \pm 1)$, M étant valeur propre de J_z. Les fonctions (5.2) deviennent:

$$\chi_{NJM}(\mathbf{x}) = \frac{1}{r} f_{NJl}(r) \cdot \Phi_{JML}(\theta, \varphi) + \frac{1}{r} f_{NJl+\varepsilon}(r) \cdot \Phi_{JMl+\varepsilon}(\theta, \varphi) , \tag{5.3}$$

où $l = J - \frac{1}{2}\varepsilon$, $\varepsilon = \pm 1$, et N est introduit pour compléter la numérotation des solutions.

Les fonction Φ_{JMl} ont les propriétés:

$$\begin{cases} a) \quad \pi\Phi_{JMl} = (-1)^l \Phi_{JMl} , & b) \quad \sigma\Phi_{JMl} = \Phi_{JMl+\varepsilon} , \\ c) \quad \mathbf{J}^2\Phi_{JMl} = J(J+1)\Phi_{JMl} , & d) \quad J_z\Phi_{JMl} = M\Phi_{JMl} , \\ e) \quad \mathbf{L}^2\Phi_{JMl} = l(l+1)\Phi_{JMl} = (J - \frac{1}{2}\varepsilon)(J - \frac{1}{2}\varepsilon + 1)\Phi_{JMl} , \\ f) \quad (\Phi_{JMl}, \Phi_{J'M'l'}) = \delta_{JJ'}\delta_{MM'}\delta_{ll'} . \end{cases} \tag{5.4}$$

Calcul de perturbation. – Par un calcul de perturbation, nous allons développer les solutions de (3.1) dans le système des solutions de l'équation

$$H_0\zeta(\mathbf{x}) = E^0\zeta(\mathbf{x}) . \tag{5.5}$$

A cause de (5.3), nous choisissons directement des ζ qui sont fonctions propres de $\mathbf{J}^2$ et J_z:

$$\zeta_{nlMJ}(\mathbf{x}) = \frac{1}{r} g_{nl}(r) \cdot \Phi_{lMJ}(\theta, \varphi) , \qquad J = l + \tfrac{1}{2}\varepsilon, \ \varepsilon = \pm 1, \tag{5.6}$$

$g_{nl}(r)$ devant satisfaire

$$\left(-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{2m}\frac{l(l+1)}{r^2} + U(r)\right) g_{nl}(r) = E^0_{nl} g_{nl}(r) . \tag{5.7}$$

Du fait qu'on néglige l'interaction électromagnétique, protons et neutrons sont traîtés symétriquement, et sont plongés dans le potentiel commun $U(r)$; ils ont donc mêmes fonctions d'onde avant l'introduction de la perturbation. Ce n'est plus le cas après, aussi introduisons-nous l'indice t d'iso-spin; par définition:

$$g_{nlt=1}(r) = \begin{pmatrix}1\\0\end{pmatrix} g_{nl}(r) \quad \text{fonction radiale à deux composantes attachée à un proton.}$$

$$g_{nlt=-1}(r) = \begin{pmatrix}0\\1\end{pmatrix} g_{nl}(r) \quad \text{fonction radiale à deux composantes attachée à un neutron.}$$

Let g_{nlt} sont fonctions propres de $\tau^3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$: $\tau^3 g_{nlt} = t g_{nlt}$. Les relations $[H', \tau^3] \neq 0$ et $[H', \mathbf{L}^2] \neq 0$ font que les χ ne peuvent être fonctions propres de τ^3 et $\mathbf{L}^2$; il est toutefois pratique de conserver la numérotation $(nlMJt)$ pour ces fonctions.

L'iso-spin et la charge ne sont pas conservés, mais tout comme la parité, ils sont « répartis » entre les nucléons et le champ mésonique par le terme $\boldsymbol{\sigma}\cdot\nabla V_\alpha(r)\tau^\alpha$.

Notons pour fixer la notation les résultats nécessaires du calcul de perturbation; en abrégeant $(nlmJt)$ par ν:

1er ordre:

$$\text{(5.8)} \qquad \begin{cases} \Delta E_\nu^{(1)} = \langle \nu | H' | \nu \rangle , \\ \beta_{\nu'\nu}^{(1)} = - \dfrac{\langle \nu' | H' | \nu \rangle}{E_{\nu'}^0 - E_\nu^0} , \end{cases} \qquad (\nu' \neq \nu),$$

2me ordre:

$$\text{(5.9)} \qquad \Delta E_\nu^{(2)} = - \sum_{\nu' \neq \nu} \frac{\langle \nu | H' | \nu' \rangle \langle \nu' | H' | \nu \rangle}{E_{\nu'}^0 - E_\nu^0} = - \sum_{\nu' \neq \nu} |\beta_{\nu'\nu}^{(1)}|^2 (E_{\nu'}^0 - E_\nu^0) .$$

Les éléments de matrice de H' sont aisément calculables:

$$\text{(5.10)} \qquad \langle \nu | H' | \nu' \rangle = \frac{f}{\mu} \langle g_{nl} \| V'_\alpha(r) \tau^\alpha \| g_{n'l't'} \rangle \cdot \langle \Phi_{lMJ} | \sigma | \Phi_{l'M'J'} \rangle =$$

$$= \delta_{JJ'} \delta_{ll'+\varepsilon'} \delta_{MM'} \cdot \frac{f}{\mu} \int_0^\infty dr\, g_{nlt}^*(r)\, V'_\alpha(r) \tau^\alpha g_{n'l+\varepsilon t'}(r) .$$

en faisant usage de (5.4 *b*) puis de (5.4 *f*).

Les seuls coefficients non nuls sont donc:

$$\text{(5.11)} \qquad \beta_{nn'}^{tt'}(l, J) = - (E_{nl}^0 - E_{n'l+\varepsilon}^0)^{-1} \langle nlMJt | H' | n'l+\varepsilon\, M\, J\, t' \rangle .$$

Ils ont la forme explicite:

$$
(5.12)\qquad
\begin{cases}
\beta^{11}_{nn'}(l,J) = -\beta^{-1-1}_{nn'}(l,J) = -\dfrac{f}{\mu}\dfrac{1}{E^0_{nl}-E^0_{n'l+\varepsilon}}\displaystyle\int_0^\infty \mathrm{d}r\, g_{nl}(r)g_{n'l+\varepsilon}(r)\,V_3'(r)\,, \\
\beta^{1-1}_{nn'}(l,J) = \quad \beta^{*-11}_{nn'}(l,J) = \\
\qquad\qquad = -\dfrac{f}{\mu}\dfrac{1}{E^0_{nl}-E^0_{n'l+\varepsilon}}\displaystyle\int_0^\infty \mathrm{d}r\, g_{nl}(r)g_{n'l+\varepsilon}(r)\big(V_1'(r)-iV_2'(r)\big)\,.
\end{cases}
$$

Les $g_{nl}(r)$ sont supposées réelles car nous utilisons dans la suite des fonctions radiales réelles.

La correction du premier ordre $\Delta E_\nu^{(1)}$ est indentiquement nulle à cause de $\langle\nu|H'|\nu\rangle = 0$; nous gardons par conséquent le second ordre $\Delta E_\nu^{(2)}$ pour l'énergie.

En résumé:

$$
(5.13)\qquad \chi_{nlMJt}(\mathbf{x}) = \frac{1}{N_{nlJt}}\cdot\{\zeta_{nlMJt}(\mathbf{x}) + \sum_{n't'}\beta^{t't}_{n'n}(l,J)\,\zeta_{n'l+\varepsilon MJt'}(\mathbf{x})\} \qquad (J = l+\tfrac{1}{2}\varepsilon)\,.
$$

La constante de normalisation $N_{nlJt} = \big(1+\sum_{n't'}|\beta^{t't}_{n'n}|^2\big)^{\frac{1}{2}}$

$$
(5.14)\qquad
\begin{cases}
\Delta E_{nlJ} = \displaystyle\sum_{n't'}|\beta^{tt'}_{nn'}(l,J)|^2(E^0_{nl}-E_{n'l+\varepsilon})\,, \\
E_{nlJ} = E^0_{nl} + \Delta E_{nlJ}\,.
\end{cases}
$$

Remarque: En examinant (5.13) on voit que pour n et J donnés il existe deux solutions, correspondant à $l = J \pm \frac{1}{2}$ (abstraction faite de M et t); ces deux solutions contiennent chacune les deux mêmes fonctions angulaires $\Phi_{l\pm\frac{1}{2},M,J}(\theta,\varphi)$. Ces paires de solutions jouent un rôle important dans les approximations futures, et correspondent à des niveaux d'énergie que nous appelons « associés ». Le caractère accessoire de numérotation de l apparaît si on compare (5.3) et (5.13): $(n,l,t)\to N$.

Construction de la fonction d'état Ψ. – Les fonctions d'onde approximatives du système nucléons-mésons ont la forme:

$$
\Psi^p = \varphi^+(X,\eta)\cdot|p\rangle + \varphi^-(\bar{X},\eta)\cdot|q\rangle\,,
$$

où $|p\rangle$ et $|q\rangle$ sont des vecteurs mésoniques de parité déterminée ($p = \pm$ et $q = \mp$); Ψ^p a la parité p.

Les fonctions $\varphi^\pm$ sont égales à:

$$
\varphi^+(X,\eta) = \tfrac{1}{2}\{\varphi(X,\eta)+\varphi(-X,\eta)\}\,, \qquad \varphi^-(X,\eta) = \tfrac{1}{2}\{\varphi(X,\eta)-\varphi(-X,\eta\}\,,
$$

où $\varphi(X,\eta)=\chi_1(\mathbf{x}_1 s_1 t_1)\dots\chi_A(\mathbf{x}_A s_A t_A)$; les χ_a sont les fonctions de quatre composantes, deux de spins et deux d'iso-spin, solutions de (3.1); notons-les $\chi_{\nu_a}(\mathbf{x}_a s_a t_a)$. La fonction Ψ^p doit être complètement antisymétrique relativement à l'échange simultané des variables d'espace, de spin et d'iso-spin de deux quelconques nucléons. Cela impose que φ^+ et φ^- le soient également, et elles le seront automatiquement si on antisymétrise φ. On redéfini donc:

$$\varphi(X,\eta)=\text{antisym.}\{\chi_{\nu_1}\dots\chi_{\nu_a}\dots\chi_{\nu_A}\}\,.$$

Si les χ_ν sont orthonormées: $(\chi_\nu,\chi_{\nu'})=\delta_{\nu\nu'}$, la fonction Ψ^p a automatiquement la norme 1:

$$(\Psi^p,\Psi^p)=\langle p\,|\,p\rangle(\varphi^+,\varphi^+)+\langle q\,|\,q\rangle(\varphi^-,\varphi^-)=\sum_\eta\int_X\varphi^*(X,\eta)\varphi(X,\eta)=1\,.$$

Car l'antisymétrisation ne change rien, et $\langle p\,|\,p\rangle=\langle q\,|\,q\rangle=1$ par hypothèse.

6. – Calcul de la source du champ V_α.

Introduisons dans ϱ^α de (2.7) les solutions (5.3), connues aux fonctions radiales près. Les sources étant additives, commençons par calculer le ϱ^α dû à une couche complète de nucléons (N,J):

$$\varrho^\alpha_{NJ}=-\frac{f}{\mu}\sum_{M=-J}^{+J}\boldsymbol{\nabla}\cdot(\chi^*_{NJM}\boldsymbol{\sigma}\tau^\alpha\chi_{NJM})= \tag{6.1}$$

$$=-\frac{f}{\mu}\sum_M\left\{\chi^*_{NJM}\tau^\alpha\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}f_{NJl}(r)\cdot\Phi_{JMl}(\theta,\varphi)+\frac{1}{r}f_{NJl+\varepsilon}(r)\cdot\Phi_{JMl+\varepsilon}(\theta,\varphi)\right)+\right.$$

$$\left.+\text{ conjugué complexe}\right\}.$$

L'opérateur $\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}$ a une action simple sur les fonctions $(1/r)f(r)\Phi_{JMl}(\theta,\varphi)$; on démontre que:

$$\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}=\sigma\left\{\frac{\partial}{\partial r}-\frac{1}{r}(\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2)\right\}, \tag{6.2}$$

d'où:

$$\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}f(r)\,\Phi_{JMl}(\theta,\varphi)\right)=\frac{1}{r}\left(\frac{\mathrm{d}f}{\mathrm{d}r}-\varepsilon\frac{J+\frac{1}{2}}{r^2}f(r)\right)\Phi_{JMl+\varepsilon}(\theta,\varphi)\,, \tag{6.3}$$

$$l=J-\tfrac{1}{2}\varepsilon\,,\qquad \varepsilon=\pm1\,,$$

en utilisant (5.4).

En appliquant (6.3), on a:

$$\varrho^{\alpha}_{NJ} = -\frac{f}{\mu}\left\{\left(\frac{1}{r^2} f^*_{NJl}\ \tau^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \varepsilon\frac{J+\frac{1}{2}}{r^2}\right) f_{NJl+\varepsilon}\cdot\sum_M \Phi^*_{JMl}\ \Phi_{JMl}\ +\right.\right.$$

$$+\frac{1}{r^2} f^*_{NJl+\varepsilon}\tau^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \varepsilon\frac{J+\frac{1}{2}}{r^2}\right) f_{NJl}\ \cdot\sum_M \Phi^*_{JMl+\varepsilon}\Phi_{JMl+\varepsilon} +$$

$$+\frac{1}{r^2} f^*_{NJl}\ \tau^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \varepsilon\frac{J+\frac{1}{2}}{r^2}\right) f_{NJl}\ \cdot\sum_M \Phi^*_{JMl}\ \Phi_{JMl+\varepsilon} +$$

$$\left.\left.+\frac{1}{r^2} f^*_{NJl+\varepsilon}\tau^{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \varepsilon\frac{J+\frac{1}{2}}{r^2}\right) f_{FJl+\varepsilon}\cdot\sum_M \Phi^*_{JMl+\varepsilon}\Phi_{JMl}\right) + \text{conj. compl.}\right\}.$$

Les sommes sur M sont des constantes:

$$(6.4)\qquad \begin{cases} \sum_{M=-J}^{+J} \Phi^*_{JMl}\Phi_{JMl} = \sum_{M=-J}^{+J} \Phi^*_{JMl+\varepsilon}\Phi_{JMl+\varepsilon} = \dfrac{J+\frac{1}{2}}{2\pi}, \\ \sum_{M=-J}^{+J} \Phi^*_{JMl}\Phi_{JMl+\varepsilon} = 0\,. \end{cases}$$

Il ne reste ainsi que des fonctions radiales:

$$(6.5)\qquad \varrho^{\alpha}_{NJ} = -\frac{f}{\mu}\frac{J+\frac{1}{2}}{2\pi}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(f^*_{NJl}(r)\tau^{\alpha}f_{NJl+\varepsilon}(r) + f^*_{NJl+\varepsilon}(r)\tau^{\alpha}f_{NJl}(r)\right).$$

La source ϱ^{α} produite par une couche complète est purement radiale; elle le reste naturellement si on additionne les contributions de N couches. Cette propriété est indispensable car nous avons postulé que $V_{\alpha} = V_{\alpha}(r)$. Les couches incomplètes engendrent un ϱ^{α} qui n'est plus de symétrie sphérique, mais elles sont en nombre très faible vis-à-vis des autres, et nous travaillons dans la suite en négligeant leur faible apport ou, ce qui est équivalent, en remplissant complètement toutes les couches. Ainsi, si i est l'indice des couches occupées:

$$(6.6)\qquad \varrho^{\alpha}(r) = \sum_i \varrho^{\alpha}_{N_iJ_i}(r)\,.$$

Propriété: Les sources dues aux protons, d'une part, et aux neutrons, d'autre part, *s'ajoutent avec le même signe.*

Dans une théorie scalaire, les sources produites par protons et neutrons se soustraient; si $\chi = \begin{pmatrix}\chi_{\mathrm{p}}\\ \chi_{\mathrm{n}}\end{pmatrix}$:

$$\varrho^2 \doteq \chi^*\tau^3\chi = \chi^*_{\mathrm{p}}\chi_{\mathrm{p}} - \chi^*_{\mathrm{n}}\chi_{\mathrm{n}} = \text{source de mésons scalaires neutres}\,.$$

Dans le cas pseudo-scalaire, il se produit le contraire; nous le démontrons pour les mésons π^0 (neutres), en supposant qu'il n'y a pas de mésons $\pi^{\pm}$

(chargés), pour simplifier. Le cas général est traité implicitement a la Section 7.

Soient $V_1 = V_2 = 0$. L'éq. (3.1) devient:

$$\left(H_0 + \frac{f}{\mu}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}V_3\tau^3\right)\chi = E\chi\,. \tag{6.7}$$

Comme $\tau^3\chi = \begin{pmatrix}\chi_p\\ -\chi_n\end{pmatrix}$, on a respectivement pour χ_p et χ_n les équations:

$$\left(H_0 + \frac{f}{\mu}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}V_3\right)\chi_p = E\chi_p\,, \tag{6.8}$$

$$\left(H_0 - \frac{f}{\mu}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}V_3\right)\chi_n = E\chi_n\,. \tag{6.9}$$

On a vu que si une fonction $\chi(\mathbf{x})$ satisfait (6.8), alors $\chi(-\mathbf{x})$ satisfait (6.9), à cause du caractère pseudo-scalaire de $V_\alpha(r)$.

D'où, réciproquement: $\chi_n(\mathbf{x}) = \chi_p(-\mathbf{x})$. Ainsi:

$$\varrho_n^3 = +\frac{f}{\mu}\boldsymbol{\nabla}(\chi_n^*\boldsymbol{\sigma}\chi_n) = -\frac{f}{\mu}\{-\boldsymbol{\nabla}(\chi_p^*(-\mathbf{x})\boldsymbol{\sigma}\chi_p(-\mathbf{x})\} = \varrho_p^3(-\mathbf{x})\,.$$

Pour une couche complète: $\varrho_p^3 = \varrho_p^3(r)$, d'où $\varrho_p^3 = \varrho_n^3$.

7. – Calcul d'un exemple.

Les potentiels pseudo-scalaires $V_\alpha(r)$ apparaissant dans (3.1) ne sont pas arbitraires, mais doivent satisfaire l'éq. (3.2) dont la source dépend des solutions de (3.1). Les sources construites avec les fonctions propres de l'hamiltonien $H_0 = -(1/2m)\Delta + U(r)$ sont identiquement nulles, d'où $V_\alpha = 0$ si on lui impose les conditions aux limites relatives au problème. C'est la solution triviale, et il faut s'assurer qu'elle n'est pas la seule. En d'autres termes: postulons dans (3.1) un certain $V_\alpha^{(0)}$; avec les solutions de cette équation on forme $\varrho_{(0)}^\alpha$ et calcule $V_\alpha^{(1)}$ satisfaisant (3.2) avec cette source. Puis, on introduit $V_\alpha^{(1)}$ dans (3.1) et recommence l'opération. Il en résulte une suite $V_\alpha^{(n)}$; si cette suite converge vers une limite V_α, et si $V_\alpha \neq 0$, c'est une autre solution, celle qui a un intérêt physique. Dans ce cas, on dit que la théorie est cohérente.

Le premier point à vérifier est l'existence générale de cette cohérence; le deuxième, que le $V_\alpha(r)$ trouvé conduit à des « splittings » des niveaux $J = l \pm \frac{1}{2}$ d'ordre de grandeur correct. On peut le faire directement sans recourir au procédé ci-dessus, en simplifiant le problème sans perdre en généralité et calculant directement la solution V_α.

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Grace à l'approximation de premier ordre du calcul de perturbation, le calcul se décompose en trois parties semblables, une pour chaque composante V_α ($\alpha = 1, 2, 3$). Supposons un instant que nous tenons compte de tous les ordres du calcul de perturbation. Les sources dépendent alors des β, ($\beta'\beta''$), etc. Pour se restreindre au premier ordre seulement, on ne conserve que les termes linéaires en β. On obtient:

$$(7.1)\quad \begin{cases} \varrho^1_{nJlt=1} = -\dfrac{f}{\mu}\dfrac{J+\frac{1}{2}}{2\pi}\dfrac{1}{r^2}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(g_{nl}(r)\sum\limits_{n'}\ (\beta^{-11}_{n'n}+\overset{*}{\beta}{}^{-11}_{n'n})g_{n'l+\varepsilon}(r)\right), \\ \varrho^2_{nJlt=1} = -\dfrac{f}{\mu}\dfrac{J+\frac{1}{2}}{2\pi}\dfrac{1}{r^2}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(g_{nl}(r)\sum\limits_{n'} i(\beta^{-11}_{n'n}-\overset{*}{\beta}{}^{-11}_{n'n})g_{n'l+\varepsilon}(r)\right), \\ \varrho^3_{nJlt=1} = -\dfrac{f}{\mu}\dfrac{J+\frac{1}{2}}{2\pi}\dfrac{1}{r^2}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(g_{nl}(r)\sum\limits_{n'}\ (\beta^{11}_{n'n}+\overset{*}{\beta}{}^{11}_{n'n})\,g_{n'l+\varepsilon}(r)\right), \end{cases}$$

sources produites par une couche $(n, J, l = J-(\varepsilon/2), t = 1)$ de protons; pour les neutrons on a:

$$(7.2)\quad \begin{cases} \varrho^1_{nJlt=-1} = -\dfrac{f}{\mu}\dfrac{J+\frac{1}{2}}{2\pi}\dfrac{1}{r^2}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(g_{nl}(r)\sum\limits_{n'}\ (\beta^{1-1}_{n'n}+\overset{*}{\beta}{}^{1-1}_{n'n})\ g_{n'l+\varepsilon}(r)\right), \\ \varrho^2_{nJlt=-1} = -\dfrac{f}{\mu}\dfrac{J+\frac{1}{2}}{2\pi}\dfrac{1}{r^2}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(g_{nl}(r)\sum\limits_{n'} i(-\beta^{1-1}_{n'n}+\overset{*}{\beta}{}^{1-1}_{n'n})\ g_{n'l+\varepsilon}(r)\right), \\ \varrho^3_{nJlt=-1} = -\dfrac{f}{\mu}\dfrac{J+\frac{1}{2}}{2\pi}\dfrac{1}{r^2}\dfrac{\mathrm{d}}{\mathrm{d}r}\left(g_{nl}(r)\sum\limits_{n'}\ (-\beta^{-1-1}_{n'n}-\overset{*}{\beta}{}^{-1-1}_{n'n})g_{n'l+\varepsilon}(r)\right). \end{cases}$$

En vertu de (5.12):

$$(\beta^{11}_{n'n}+\overset{*}{\beta}{}^{11}_{n'n}) = -(\beta^{-1-1}_{n'n1}+\overset{*}{\beta}{}^{-1-1}_{n'n}) = \text{fonctionelle de } V_3(r),$$
$$(\beta^{-11}_{n'n}+\overset{*}{\beta}{}^{-11}_{n'n}) = (\beta^{1-1}_{n'n}+\overset{*}{\beta}{}^{1-1}_{n'n}) = \text{fonctionelle de } V_1(r),$$
$$i(\beta^{-11}_{n'n}-\overset{*}{\beta}{}^{-11}_{n'n}) = i(\beta^{1-1}_{n'n}+\overset{*}{\beta}{}^{1-1}_{n'n}) = \text{fonctionelle de } V_2(r)\ (*).$$

Posons:

$$\xi^3_{n'n} = \frac{1}{2}(\beta^{11}_{n'n}+\overset{*}{\beta}{}^{11}_{n'n}), \qquad \xi^2_{n'n} = \frac{i}{2}(\beta^{-11}_{n'n}-\overset{*}{\beta}{}^{-11}_{n'n}), \qquad \xi^1_{n'n} = (\beta^{-11}_{n'n}+\overset{*}{\beta}{}^{-11}_{n'n}).$$

(*) Ce résultat montre que les sources ϱ^α dues aux protons et aux neutrons sont identiques, non seulement pour les mésons π^0 comme nous l'avons démontré à la Sect. 6 mais aussi pour les mésons chargés.

Les $\xi^{\alpha}_{n'n}$ dépendent de J et l, naturellement; ils valent:

$$\xi^{\alpha}_{n'n}(l, J) = -\frac{f}{\mu}\frac{1}{E^0_{n'l+\varepsilon} - E^0_{nl}} \cdot \int_0^{\infty} \mathrm{d}r\, g_{nl}(r) g_{n'l+\varepsilon}(r)\, V'_{\alpha}(r)\,. \tag{7.3}$$

Solutions de (3.1):

$$\chi_{nlM\,Jt=1}(\mathbf{x}) = \begin{pmatrix}1\\0\end{pmatrix}\frac{1}{r} g_{nl}(r)\Phi_{lM\,J}(\theta, \varphi) + \sum_{n'}\left\{\begin{pmatrix}\xi^3_{n'n}\\0\end{pmatrix} + \begin{pmatrix}0\\\xi^1_{n'n}\end{pmatrix} + \right.$$
$$\left. + \begin{pmatrix}0\\\xi^2_{n'n}\end{pmatrix}\right\}\frac{1}{r} g_{n'l+\varepsilon}(r)\Phi_{l+\varepsilon M\,J}(\theta, \varphi)\,.$$

Equation pour $V_{\alpha}(r)$:

$$(\Delta - \mu^2) V_{\alpha} = -\frac{f}{\mu}\sum_{nlJt}\frac{J+\frac{1}{2}}{2\pi}\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(g_{nl}(r)\sum_{n'}\xi^{\alpha}_{n'n}(l, J) g_{n'l+\varepsilon}(r)\right). \tag{7.4}$$

Les fonctions χ sont linéaires en ξ^{α}, et la source de l'équation régissant $V_{\alpha'}$ ne contient que des $\xi^{\alpha'}$. Dans l'approximation du premier ordre le problème se découple, et on peut traiter séparément les calculs relatifs aux trois potentiels $V_{\alpha=1,2,3}$. La théorie sera cohérente s'il existe des $\xi^{\alpha}_{n'n} \neq 0$, calculés par (7.3) et tels que (7.4) soit satisfaite. On voit que les calculs sont parfaitement les mêmes pour $\alpha = 1$, 2 ou 3 (en conséquence, nous sous-entendrons dorénavant l'indice α). De plus les effets dans le splitting sont additifs; il suffit donc de multiplier par trois le ΔE^{α} trouvé dans un calcul pour avoir le splitting total.

Le membre de droite de (7.4) contient une infinité de $\xi^{\alpha}_{n'n}$; on peut en limiter le nombre, et rendre ainsi le calcul praticable. Les $\xi_{n'n}$ sont des éléments de matrice (7.3) qui décroissent rapidement lorsque croît l'énergie séparant les états entre lesquels ils sont calculés. Nous n'en conservons qu'un (à une exception près), celui pour lequel $|E^0_{n'l+\varepsilon} - E^0_{nl}| =$ minimum, n' variant (dans la numérotation habituelle, c'est pour $n' = n$). Il s'ensuit une compensation considérable dans les sources; considérons les ϱ engendrés par les couches (n, l, J) et $(n, l+\varepsilon, J)$, dites associées (*):

$$\varrho_{nlJ} \simeq -\frac{f}{\mu}\frac{J+\frac{1}{2}}{2\pi}\xi_{nn}(l, J)\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(g_{nl}g_{nl+\varepsilon})\,,$$

$$\varrho_{nl+\varepsilon J} \simeq -\frac{f}{\mu}\frac{J+\frac{1}{2}}{2\pi}\xi_{nn}(l+\varepsilon, J)\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(g_{nl}g_{nl+\varepsilon})\,.$$

(*) Voir remarque, Sect. 5.

On voit d'après (7.3) que $\xi_{nn}(l, J) = -\xi_{nn}(l+\varepsilon, J)$, donc la somme des deux ϱ est nulle dans l'approximation faite. Les seules contributions qui restent dans le ϱ_{tot} sont celles des couches près du niveau de Fermi qui sont associées à des niveaux non perturbés vides.

La conséquence est qu'on peut ramener à un nombre très restreint le nombre des coefficients $\xi_{n'n}$ à déterminer. L'approximation que nous utilisons pour cela est assez radicale, mais pour calculer un ordre de grandeur de « splitting » et vérifier l'existence d'une solution, elle est suffisante.

Données de l'exemple. – Nous prenons un noyau constitué de $A = 120$ nucléons ($N = 70$ et $Z = 50$); le potentiel $U(r)$ introduit est celui de l'oscillateur harmonique, qui permet de trouver une solution exacte

$$U(r) = U_0 \left\{ \left(\frac{r}{a}\right)^2 - 1 \right\}.$$

U_0 = profondeur du potentiel = 59 MeV;

a = rayon du noyau = $7.76 \cdot 10^{-13}$ cm.

Les fonctions radiales $g_{nl}(r)$ du système non perturbé doivent satisfaire:

$$\left\{ -\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{2m}\frac{l(l+1)}{r^2} + U_0 \cdot \left(\left(\frac{r}{a}\right)^2 - 1 \right) \right\} g_{nl}(r) = E^0_{nl} g_{nl}(r) .$$

La solution est bien connue:

$$g_{nl}(r) = \sqrt{\frac{p}{a}}\, \gamma_{nl} z^{-l/2} \exp[z/2] \frac{\mathrm{d}^k}{\mathrm{d}z^k} \left(z^{k+l+\frac{1}{2}} \exp[-z] \right) , \tag{7.5}$$

avec:

$$k = \frac{1}{2}(n - l - 1) > 0 , \qquad z = \left(p\,\frac{r}{a}\right)^2 , \qquad \gamma_{nl} = \sqrt{2}[\Gamma(k + l + \tfrac{3}{2})\Gamma(k+1)]^{-\frac{1}{2}} ,$$

$$E^0_{nl} = -U_0 \cdot \left(1 - \frac{2n+1}{p^2}\right) , \qquad p^4 = 2ma^2 U_0 .$$

En tenant compte de la position qu'auraient les niveaux dans un potentiel en forme de cuvette et de celle qu'ils auront après la perturbation, le schéma des niveaux remplis s'établit comme indiqué à la Fig. 1.

Les sources engendrées par les niveaux associés (reliés par une ligne sur la figure) se compensent en première approximation. Parmi ceux qui restent (indiqués $\nearrow s$) nous négligeons ceux de spin $\frac{1}{2}$ et $\frac{3}{2}$ car $\varrho \div J + \frac{1}{2}$. Nous retenons

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finalement la source:

$$\varrho(r) = 2 \cdot \varrho_{5,4,\frac{9}{2}}(r) + \varrho_{5,2,\frac{5}{2}}(r) + \varrho_{4,3,\frac{7}{2}}(r) \,. \tag{7.6}$$

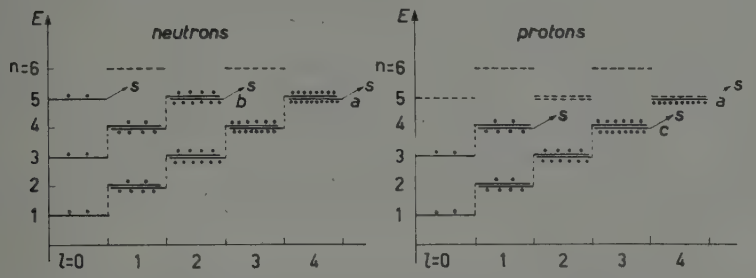

Fig. 1. – Schéma de remplissage des niveaux non perturbés. Les couches notées *a*, *b*, *c* sont retenues pour le calcul des sources.

Fonctions d'onde des particules participant aux sources:

$$\left\{\begin{aligned}
&\text{Niveau } a \ (n=5,\ l=4,\ J=\tfrac{9}{2}):\\
&\chi_a(M) = \frac{1}{N_a}\left(\frac{1}{r}\, g_{5,4}(r)\, \Phi_{4,M,\frac{9}{2}}(\theta,\varphi) + \xi_1 \frac{1}{r}\, g_{6,5}(r)\, \Phi_{5,M,\frac{9}{2}}(\theta,\varphi)\right).\\
&\text{Niveau } b \ (n=5,\ l=2,\ J=\tfrac{5}{2}):\\
&\chi_b(M) = \frac{1}{N_b}\left(\frac{1}{r}\, g_{5,2}\cdot \Phi_{2,M,\frac{5}{2}} + \frac{1}{r}\,(-\xi_2 g_{6,3} + \xi_3 g_{4,3})\cdot \Phi_{3,M\frac{5}{2}}\right.\\
&\text{Niveau } c \ (n=4,\ l=3,\ J=\tfrac{7}{2}):\\
&\chi_c(M) = \frac{1}{N_c}\left(\frac{1}{r}\, g_{4,3}\cdot \Phi_{3,M,\frac{7}{2}} + \xi_4 \frac{1}{r}\, g_{5,4}\cdot \Phi_{4,M,\frac{7}{2}}\right).
\end{aligned}\right. \tag{7.7}$$

avec: $N_a = \sqrt{1+(\xi_1)^2}$, $N_b = \sqrt{1+(\xi_2)^2+(\xi_3)^2}$, $N_c = \sqrt{1+(\xi_4)^2}$, les fonctions n'étant pas automatiquement normalisées car on a coupé dans le développement en série. Pour simplifier la notation, nous avons posé:

$$\xi_1 = \xi_{6,5}(4,\tfrac{9}{2})\,, \quad \xi_2 = -\,\xi_{4,5}(2,\tfrac{5}{2})\,, \quad \xi_3 = \xi_{6,5}(2,\tfrac{5}{2})\,, \quad \xi_4 = \xi_{5,4}(3,\tfrac{7}{2})\,. \tag{7.8}$$

Le $\xi_{n'n}$ suivant $\xi_{6,5}(4,\frac{9}{2})$, par exemple, est $\xi_{8,5}$; il est proportionnel à $(E^0_{8,5} - E^0_{5,4})^{-1} = \frac{1}{3}\,(E^0_{6,5} - E^0_{5,4})^{-1}$, donc au moins trois fois plus petit que ξ_1. Sa contribution au ΔE est $\sim \frac{1}{9}$ de celle de ξ_1, on peut donc le négliger en première approximation.

Par contre, ξ_2 et ξ_3 sont du même ordre du grandeur puisqu'il y a un niveau $(n=6,\ l=3)$ en dessus et un niveau $(n=4,\ l=3)$ en dessous de b $(n=5,\ l=2)$, tous deux à la même distance. Il faut conserver les deux coefficients.

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En vertu de ce qui précède ((7.4) et suite) nous ne différencions pas explicitement neutrons et protons, et sous-entendons dans les χ les symboles $\begin{pmatrix}1\\0\end{pmatrix}$ ou $\begin{pmatrix}0\\1\end{pmatrix}$, car de toute façon les sources sont identiques. En introduisant (7.7) dans (7.6), on a:

$$\varrho(r) = -\frac{f}{\mu}\left\{\frac{10}{\pi} Y_1 \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(g_{5,4}g_{6,5}) + \frac{4}{\pi} Y_4 \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(g_{4,3}g_{5,4}) + \right. \tag{7.9}$$

$$\left. + \frac{3}{\pi}\left(- Y_2 \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(g_{5,2}g_{4,3}) + Y_3 \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(g_{5,2}g_{6,3})\right)\right\},$$

où

$$Y_1 = \frac{\xi_1}{N_a^2} = \frac{\xi_1}{1+(\xi_1)^2}, \qquad Y_2 = \frac{\xi_2}{1+(\xi_2)^2+(\xi_3)^2},$$

$$Y_3 = \frac{\xi_3}{1+(\xi_2)^2+(\xi_3)^2}, \qquad Y_4 = \frac{\xi_4}{1+(\xi_4)^2}.$$

Résolution. – Il faut trouver les ξ_i tels que:

$$(\Delta - \mu^2) V(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}V}{\mathrm{d}r}\right) - \mu^2 V = \varrho(r), \tag{7.10}$$

avec le $\varrho(r)$ donné en (7.9), et, selon (7.3):

$$\xi_i = -\frac{f}{\mu}\frac{p^2}{2U_0}\int_0^\infty \mathrm{d}r\, G_i(r)\, V'(r), \tag{7.11}$$

$$G_1(r) = g_{6,5}g_{5,4}, \quad G_2(r) = g_{4,3}g_{5,2}, \quad G_3(r) = g_{6,3}g_{5,2}, \quad G(r) = g_{5,4}g_{4,3}.$$

A) Calcul de $V(r)$ en fonction des ξ. Posons $x = p(r/a)$, $\lambda = \mu a/p$, et changeons de variables dans (7.10); il vient:

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}V}{\mathrm{d}x}\right) - \lambda^2 V = \left(\frac{a}{p}\right)^2 \varrho.$$

En portant dans (7.9) les formes explicites des g_{nl}, introduisant la nouvelle variable x et groupant les termes de même puissance on obtient:

$$\varrho(x) = -\frac{f}{\mu}\left(\frac{p}{a}\right)^3 \left\{\left(\frac{10}{\pi}\gamma_1 Y_1 + \frac{3}{\pi}\gamma_3 Y_3\right)\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(x^{11}\exp[-x^2]) + \right. \tag{7.12}$$

$$+ \left(\frac{4}{\pi}\gamma_4 Y_4 + \frac{3}{\pi}\gamma_2 Y_2 - \frac{24}{\pi}\gamma_3 Y_3\right)\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(x^9\exp[-x^2]) +$$

$$\left. + \left(-\frac{21}{2\pi}\gamma_2 Y_2 + \frac{189}{4\pi}\gamma_3 Y_3\right)\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(x^7\exp[-x^2])\right\},$$

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où

$$\gamma_1 = 2\left(\frac{11}{2}!\,\frac{9}{2}!\right)^{-\frac{1}{2}}, \qquad \gamma_2 = 2\left(\frac{7}{2}!\right)^{-1}, \qquad \gamma_3 = \gamma_4 = 2\left(\frac{7}{2}!\,\frac{9}{2}!\right)^{-\frac{1}{2}}.$$

L'équation différentielle est linéaire, ce qui entraîne que

$$V = \frac{f}{\lambda}\left(\frac{p}{a}\right)\sum_{\nu=0}^{2} c_\nu v_\nu(x)\,, \tag{7.13}$$

avec v_ν satisfaisant:

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}v_\nu}{\mathrm{d}x}\right) - \lambda^2 v_\nu = -\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{7+2\nu}\exp\left[-x^2\right]\right). \tag{7.14}$$

L'éq. (7.14), en posant $v_\nu = (1/x)w_\nu$, se ramène à

$$w_\nu''(x) - \lambda^2 w_\nu(x) = -\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{7+2\nu}\exp\left[-x^2\right]\right),$$

qui se résout par la méthode de Green. En exigeant que v_ν soit continu à l'origine ainsi que sa dérivée, et que $v_\nu \to 0$ $(x \to \infty)$, on a:

$$v_\nu(x) = \frac{1}{x}\left\{\exp\left[-\lambda x\right]\int_0^x \mathrm{d}t\,\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{7+2\nu}\exp\left[-t^2\right]\right)\sinh\lambda t + \right. \tag{7.15}$$

$$\left. + \sinh\lambda x\int_x^\infty \mathrm{d}t\,\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{7+2\nu}\exp\left[-t^2\right]\right)\exp\left[-\lambda t\right]\right\}.$$

Les intégrales donnant v_ν sont des cas particuliers de:

$$M(\zeta, \eta, \alpha, n) = \int_\eta^\zeta \mathrm{d}t(nt^{n-2} - 2t^n)\exp\left[-(t+\alpha)^2\right],$$

qui s'intègre par partie, à une fonction d'erreur près. Mais le plus simple est de poser:

$$M(\zeta, \eta, \alpha, n) = \left|P_n(\alpha, t)\exp\left[-(t+\alpha)^2\right]\right|_\eta^\zeta - b_n(\alpha)\int_\eta^\zeta \mathrm{d}t\exp\left[-(t+\alpha)^2\right],$$

avec $P_n(\alpha, t) = (\text{polynôme en } \alpha \text{ et } t) = \sum_m A_m^n(\alpha)t^m$.

En dérivant M, on a:

$$\sum_m \{(m+1)A_{m+1} - 2\alpha A_m - 2A_{m-1}\}t^m - b_n(\alpha) = nt^{n-1} - t^n ,$$

d'où on tire une relation de récurence pour les $A_m^n(\alpha)$.

Cette méthode a l'avantage d'être applicable à d'autres exemples sans changement.

On trouve finalement:

$$(7.16) \qquad v_\nu(x) = -\frac{1}{\lambda}\frac{1}{x} q_{7+2\nu}(\lambda/2, x) \exp[-x^2] +$$

$$+ \frac{\sqrt{\pi}}{4\lambda} b_{7+2\nu}(\lambda/2) \frac{1}{x} (\exp[\lambda x][E(x+\lambda/2) - 1] + \exp[-\lambda x][E(x-\lambda/2)+1]) ,$$

où

$$q_{7+2\nu}(\lambda/2, x) = \sum_{m=0}^{\nu+2} A_{2m+1}^{7+2\nu}(\lambda/2) \cdot x^{2m+1} ,$$

$A_{2m+1}^{7+2\nu}(\lambda/2)$ = coefficients dépendant de $\lambda = \mu a/p$, calculés à partir de la relation de récurence,

$$E(x \pm \lambda/2) = \frac{2}{\sqrt{\pi}} \int_0^{x \pm \lambda/2} \exp[-t^2]\, dt . \qquad \text{(Jahnke et Emde, fonction d'erreur),}$$

(voir Fig. 2).

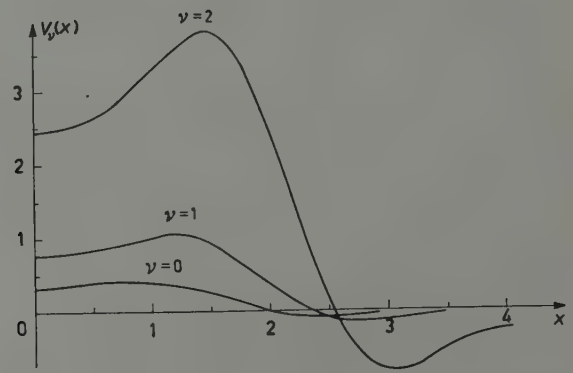

Fig. 2. – Fonctions $v_\nu(x)$, $\nu = 0, 1, 2$ (formule (7.16)).

Et $V(r)$ explicitement en fonction des ξ_i, par l'intérmédiaire des Y_i (7.9):

$$(7.17) \qquad V(r) = \frac{f}{\lambda}\frac{p}{a}\sum_{\nu=0}^{2} c_\nu(\xi) v_\nu(x) = \frac{f}{\lambda}\frac{p}{a}\left\{\left(\frac{10}{\pi}\gamma_1 Y_1 + \frac{3}{\pi}\gamma_3 Y_3\right)\cdot v_2(x) + \right.$$

$$\left. + \left(\frac{3}{\pi}\gamma_2 Y_2 - \frac{24}{\pi}\gamma_3 Y_3 + \frac{4}{\pi}\gamma_4 Y_4\right)\cdot v_1(x) + \left(-\frac{21}{2\pi}\gamma_2 Y_2 + \frac{189}{4\pi}\gamma_3 Y_3\right)\cdot v_0(x)\right\} .$$

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B) Calcul des ξ_i. En portant dans la définition des ξ_i l'expression de $V(r)$ calculée ci-dessus, on obtient quatre équations algébriques pour les quatre ξ_i inconnus:

$$\xi_i = -\frac{f}{\mu}\frac{p^2}{2U_0}\int_0^\infty \mathrm{d}r\, G_i(r)\, V'(r) = \frac{f}{\mu}\frac{p^3}{2U_0}\int_0^\infty \mathrm{d}r\, G_i'(r)\, V(r) =$$

$$= \left(\frac{f}{\mu}\right)^2 \frac{p^3}{2aU_0}\,\gamma_i \sum_{\nu=0}^{2} c_\nu(\xi)\int_0^\infty \mathrm{d}x\, v_\nu(x)\,\frac{\mathrm{d}G_i}{\mathrm{d}x}\,.$$

Posons:

$$J_\nu^i = \int_0^\infty \mathrm{d}x\, v_\nu(x)\,\frac{\mathrm{d}G_i}{\mathrm{d}x}\,. \tag{7.18}$$

Il vient au total:

$$\xi_i = f^2\,\frac{p^3}{\lambda^2\cdot 2aU_0}\,\gamma_i \sum_{\nu=0}^{2} c_\nu(\xi)\,J_\nu^i\,. \tag{7.19}$$

Toutes les grandeurs figurant dans ces équations sont connues; on calcule graphiquement les J_ν^i, et laisse pour l'instant le choix de f ouvert. En posant

$$\alpha_1^i = \frac{f^2p^3}{\lambda^2\cdot 2aU_0}\,\gamma_1\gamma_i\left(\frac{10}{\pi}J_0^i\right), \qquad \alpha_4^i = \frac{f^2p^3}{\lambda^2\cdot 2aU_0}\,\gamma_4\gamma_i\left(\frac{4}{\pi}J_1^i\right),$$

$$\alpha_2^i = \frac{f^2p^3}{\lambda^2\cdot 2aU_0}\,\gamma_2\gamma_i\left(-\frac{3}{\pi}J_1^i + \frac{21}{2\pi}J_2^i\right), \quad \alpha_3^i = \frac{f^2p^3}{\lambda^2\cdot 2aU_0}\,\gamma_3\gamma_i\left(\frac{3}{\pi}J_0^i - \frac{24}{\pi}J_1^i + \frac{189}{4\pi}J_2^i\right)$$

(7.19) s'écrit:

$$\xi_i = \sum_{k=1}^{4} \alpha_k^i\, Y_k(\xi)\,. \tag{7.19'}$$

Les α_k^i sont des coefficients numériques, et les $Y_k(\xi)$ les grandeurs définies en (7.19). (7.19') est une équation non linéaire à quatre inconnues ξ.

La solution est, pour $f = 1.1$ (*):

$$\left\{\begin{array}{l} \xi_1 = 0.312\,, \quad \xi_2 = 0.050\,, \quad \xi_3 = 0.680\,, \quad \xi_4 = 0.285\,, \\ \text{d'où:} \\ Y_1 = 0.284\,, \quad Y_2 = 0.034\,, \quad Y_3 = 0.464\,, \quad Y_4 = 0.264\,. \end{array}\right. \tag{7.20}$$

(*) Moyenne des valeurs généralement admises pour f.

C) Forme explicite de $V(r)$. En remplaçant les Y_i dans (7.17) par leur valeur, on trouve:

$$V(r) = 3.82 \cdot v_2(x) - 19.2 \cdot v_1(x) + 42.7 \cdot v_0(x) \text{ (MeV)} . \tag{7.21}$$

(Voir Fig. 3).

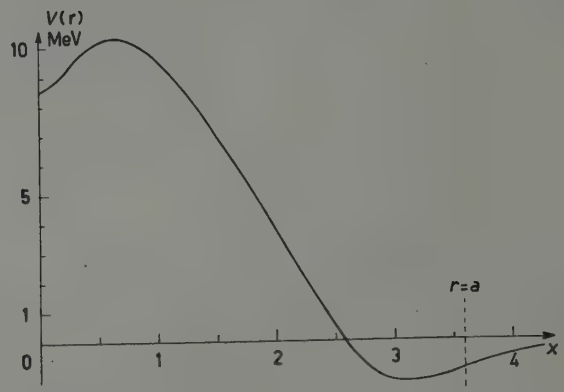

Fig. 3. – Potentiel pseudo-scalaire $V(r)$, $r = x \cdot a/p$ (formule (7.21)).

Explicitement:

$$V(r) = \frac{f}{\lambda}\frac{p}{a}\left\{Q(x)\exp[-x^2] + \right.$$
$$\left. + B\frac{1}{x}(\exp[\lambda x][E(x+\lambda/2)-1] + \exp[-\lambda x][E(x-\lambda/2)+1])\right\},$$
$$Q(x) = 2.15 + 0.88x^2 + 0.26x^4 - 0.013x^6 + 0.25x^8,$$
$$x = p\cdot\frac{r}{a}, \qquad p = 3.6, \qquad \lambda = 1.46, \qquad B = 4.65 .$$

Discussion de la cohérence. – La condition physique de cohérence est:

$$\tilde{V} = V , \tag{7.22}$$

où V est un potentiel postulé, et $\tilde{V}$ le nouveau potentiel, calculé à partir des sources dépendant de V. Explicitons cette relation:

$$\sum_{\nu=0}^{2} \tilde{c}_\nu v_\nu(x) = \sum_{\nu=0}^{2} c_\nu v_\nu(x) .$$

D'où

$$\tilde{c}_\nu = c_\nu(\tilde{\xi}) = c_\nu(\xi) . \tag{7.23}$$

Ces trois relations relient les anciens ξ_i $(i=1, ..., 4)$ et les nouveaux $\tilde{\xi}$, qui d'autre part sont définis par:

$$\tilde{\xi}_i = \sum_{k=1}^{4} \alpha_k^i Y_k(\xi) , \tag{7.24}$$

les α_k^i étant les coefficients de l'éq. (7.19′).

(7.23) est évidemment satisfaite si on pose $\tilde{\xi}_i = \xi_i$, ce qui, avec (7.24), donne l'éq. (7.19′). Par contre, si on substitue (7.24) dans le membre de gauche de (7.23), on a trois relations pour les quatres inconnues ξ_i: le problème a l'air indéterminé. Ce n'est pourtant pas le cas car $\det|\alpha_k^i| = 0$, ce qui entraîne une dépendence linéaire des ξ_i, ou des $\tilde{\xi}_i$. Il ne reste alors que trois inconnues dans l'éq. (7.23); (7.22) et (7.19′) sont bien des conditions équivalentes. La dépendence linéaire des ξ_i est évidente si on considère (7.11): les quatre produits $G_i(r)$ ne sont pas linéairement indépendants. Cela provient de ce que deux fonctions $g_{nl}(r)$ et $g_{n'l'}(r)$ ne sont orthogonales pour $n \neq n'$ que si $l = l'$. Il y a exactement autant de $\xi_{n'n}$ linéairement indépendants dans le cas général qu'il y a de produits $g_{n'l+\varepsilon}(r) \cdot g_{nl}(r)$ linéairement indépendants. Ainsi, dans tout problème analogue au présent, on aura une dépendance linéaire entre les $\xi_{n'n}$, à condition de prendre garde en approximant de ne pas négliger trop de coefficients. Ce fait important permet de tirer quelques conclusions sur l'existence d'une solution cohérente dans un cas quelconque.

Partons d'un cas à deux coefficients:

$$\tilde{\xi}_i = \sum_{k=1}^{2} \alpha_k^i Y_k , \qquad \det|\alpha_k^i| \neq 0 , \qquad i = 1, 2.$$

On veut $\tilde{\xi}_i = \xi_i$; la solution se trouve à l'intersection des courbes $\xi_1 = \sum_k \alpha_k^1 Y_k$ et $\xi_2 = \sum_k \alpha_k^2 Y_k$. Ces courbes ne se coupent qu'à l'origine, sauf cas très particuliers (Fig. 4). On a donc presque toujours, dans ce cas, uniquement la solution triviale; il en va de même si $i = 1, 2, ..., n > 2$.

Prenons maintenant $i = 1, 2, 3$:

$$\xi_i = \sum_{k=1}^{3} \alpha_k^i Y_k , \tag{7.25}$$

avec $\det|\alpha_k^i| = 0$, qui entraîne

$$\xi_3 = b_1 \xi_1 + b_2 \xi_2 . \tag{7.26}$$

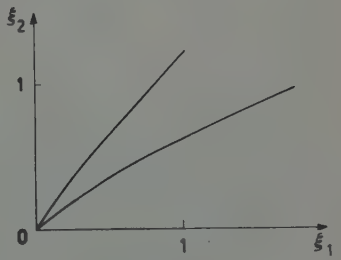

Fig. 4.

On débute avec:

$$(7.27)\qquad \begin{cases} \xi_1 = \alpha_3^1 Y_3 + \sum_{k=1}^{2} \alpha_k^1 Y_k\,, \\ \xi_2 = \alpha_3^2 Y_3 + \sum_{k=1}^{2} \alpha_k^2 Y_k\,, \end{cases}$$

et résout en gardant Y_3 arbitraire (Fig. 5).

Le lieu des solutions, noté en pointillé, va de 0 à une valeur < 1 (*) lorsque Y_3 varie de 0 à $\frac{1}{2}$, qui est sa valeur maximum (Fig. 6). La solution

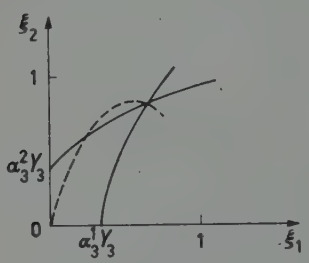

Fig. 5.

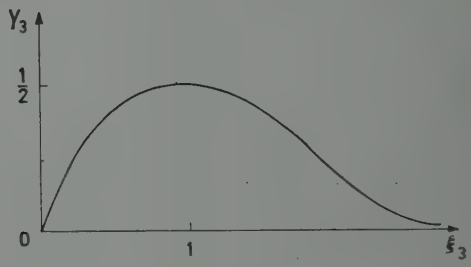

Fig. 6.

de (7.27) se trouve à l'intersection des courbes $f_1(\xi_3) = b_1 \cdot \xi_1(\xi_3) + b_2 \cdot \xi_2(\xi_3)$ et $f_2(\xi_3) = \xi_3$ (Fig. 7).

Le domaine des solutions est maintenant beaucoup plus large. Il en existe une dès que $df_1/d\xi_3|_{\xi_3=0} > 1$.

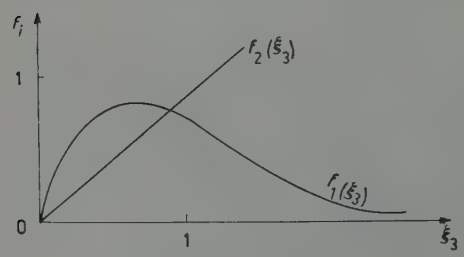

Fig. 7.

(*Remarque*: Le signe des α_k^i joue peu de rôle, car toutes les expressions ont une symétrie centrale.)

Il en va de même lorsqu'on a plus de trois coefficients, mais avec des hypersurfaces.

En résumé, nous voyons que la cohérence du problème n'existe pas par coïncidence, et pour une valeur déterminée de la constante de couplage f. Les quelques considérations faites ci-dessus sur les systèmes d'équations intervenant montrent qu'en général l'existence d'une solution n'est pas du tout problématique, comme ce serait le cas si $\det|\alpha_k^i| \neq 0$. On peut voir de même que les solutions tendent vers zero si $f \to 0$.

(*) Les α_k^i dépassent rarement 1 en valeur absolue, et seulement de peu.

Splittings. – Il n'est pas intéressant de calculer les splittings de tous les niveaux dans le présent exemple. Le potentiel $U(r)$ choisi est trop loin des potentiels réels en forme de cuvette, et il donne lieu à des fonctions radiales $g_{nl}(r)$ trop ramassées vers l'origine. D'autre part, nous avons négligé des sources provenant des niveaux à petits l, étant donné leur moindre importance quantitative. On obtient ainsi un $V(r)$ de pente plus forte vers l'origine, ce qui favorise les splittings pour l petit. Le fait important est que les splittings ont un ordre de grandeur satisfaisant; on trouve, par exemple, pour $n=5$, $l=4$, $J=4\pm\frac{1}{2}$:

$$\Delta E \simeq 1.5 \text{ MeV} .$$

En multipliant par trois, pour tenir compte des contributions des trois types de mésons, on arrive à:

$$\Delta E_{\text{tot}} \simeq 4.5 \text{ MeV} .$$

Le calcul de ce ΔE fournit une justification d'une des approximations que nous avons faites.

$$\Delta E = \Delta E_{5,4,\frac{7}{2}} - \Delta E_{5,4,\frac{9}{2}} = \text{différence des déplacements} .$$

Par exemple:

$$\Delta E_{5,4,\frac{7}{2}} = -\sum |\xi_{n'n}(4,\tfrac{7}{2})|^2 (E^0_{n',3} - E^0_{5,4}) = (\xi_4)^2 \cdot 9 - (\xi_{6,5})^2 \cdot 9 - (\xi_{8,5})^2 \cdot 27 - \ldots =$$

$$= 0.81 - 0.18 - 0.06 - \ldots \simeq 0.6 \text{ MeV} .$$

$$(\xi_{6,5})^2 \simeq 0.2\,(\xi_4)^2 , \qquad (\xi_{8,5})^2 \simeq 0.02\,(\xi_4)^2 .$$

Les termes négligés pour le calcul de $V(r)$, comme $\xi_{6,5}$, $\xi_{8,5}$, sont nettement plus petits que les termes retenus, et particulièrement ceux contenant au dénominateur une différence $|E^0_{n'l+\varepsilon} - E^0_{nl}|$ plus grande que 9 MeV (qui est le minimum).

8. – Niveaux d'énergie.

Du point de vue du spectre d'énergie, le présent modèle pseudo-scalaire est semblable au modèle en couches avec couplage spin-orbite. On peut prendre le même potentiel scalaire $U(r)$ dans les deux cas, ce qui donne la même répartition des niveaux. A part cela, le terme $\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} V_\alpha \tau^\alpha$ lève la dégénérescence en J comme le terme $\mathbf{L}\cdot\mathbf{S}$, de façon que le niveau $J=l+\frac{1}{2}$ se trouve en-

dessous de celui à $J=l-\frac{1}{2}$; en effet, les déplacements des niveaux valant:

$$(8.1)\qquad \Delta E_{nlJ=l\mp\frac{1}{2}} = -\sum_{\substack{n'\\ \alpha=1,2,3}} |\xi^{\alpha}_{n'n}(l,J)|^2 (E^0_{n'l\pm1}-E^0_{nl}) \simeq -3\,|\xi_{n''n}|^2 (E^0_{n''l\pm1}-E^0_{nl})\,,$$

et étant donné que:

$$(E^0_{n''l+1}-E^0_{nl})>0\,,\qquad (E^0_{n''l-1}-E^0_{nl})<0\,,$$

on a $\Delta E_{nlJ=l+\frac{1}{2}}<0$, $\Delta E_{nlJ=l-\frac{1}{2}}>0$, et $E_{nlJ=l+\frac{1}{2}}<E_{nlJ=l-\frac{1}{2}}$.

L'indice n'' désigne les deux niveaux $(n', l\pm1)$ les plus proches du niveau (n, l) avant perturbation (avec un $U(r)$ en forme de cuvette, on a $n''=n$).

Les $\xi_{n,n'\neq n''}$ sont suffisamment plus petits que $\xi_{nn''}$ pour que le résultat ne soit pas modifié si on les fait intervenir. L'exemple calculé à la Section **7** le montre, et dans un cas où le potentiel $U(r)$ défavorise l'approximation en rendant équidistants les niveaux non perturbés. L'ordre de grandeur du splitting que nous obtenons est de quelques MeV en moyenne, ce qui correspond aux valeurs observées. La question qui reste ouverte est la variation du splitting en fonction des nombres quantiques n et l. Il est très difficile d'y répondre en général, car la variation des ΔE à travers du spectre d'énergie dépend considérablement du potentiel $U(r)$. Il est tout au plus possible d'avancer une remarque spéculative.

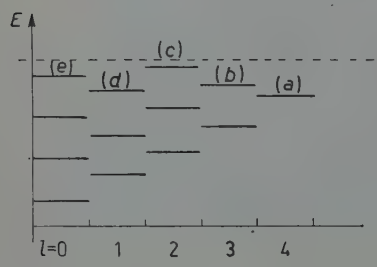

Fig. 8. – La ligne pointillée désigne la limite de Fermi.

Considérons les niveaux les plus proches de la limite de Fermi, de différents l, numérotés (a), (b), (c), etc., dans la Fig. 8. Les déplacements ont la forme (8.1). Pour (a), toutes les différences $E^0_{n'l+1}-E^0_{nl}$ ont le même signe car aucun niveau $(n', l+1)$ ne se trouve en dessous de lui. Pour (b), (c), etc., on a des termes de signes différents, d'où compensation d'autant plus grande qu'il y a de niveaux en dessous du niveau considéré. Et de même pour $(l-1)$. Tout dépend de l'importance des termes $\xi_{nn'\neq n''}$ par rapport à la variation de $\xi_{nn''}$ en fonction de l, mais il est probable qu'on obtient en général une diminution du splitting avec l au voisinage de la ligne de Fermi. Et ce sont là les seuls splittings mesurables.

9. – Parité.

Etant donné une solution $\varphi(X,\eta)=\varphi^+(X,\eta)+\varphi^-(X,\eta)$ de l'éq. (3.1'), l'état fondamental est de parité (+) et est représenté par $\Psi^+=\varphi^+|+\rangle+\varphi^-|-\rangle$, lorsque $\Delta E>0$; il est de parité (—) et représenté par $\Psi^-=\varphi^+|-\rangle+\varphi^-|+\rangle$,

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quand $\Delta E < 0$. Nous allons montrer que la parité ainsi définie correspond à la parité donnée par le modèle en couches ordinaire.

Le ΔE est donné par la formule (2.8). Il est démontré à la fin du paragraphe que le second terme est nul; il ne reste que:

$$\Delta E = h\{(\varphi^+, \varphi^+) - (\varphi^-, \varphi^-)\}. \tag{9.1}$$

Le facteur h ne concerne que les mésons, et $(\varphi^+, \varphi^+) - (\varphi^-, \varphi^-)$ seulement les nucléons. Calculons d'abord l'ordre de grandeur des produits scalaires $N^\pm = (\varphi^\pm, \varphi^\pm)$. Les fonctions $\varphi^\pm$ sont des sommes de produits de fonctions relatives à une particule, du type:

$$\chi_{nlMJt} \simeq \frac{1}{r} g_{nlt}(r)\, \Phi_{lMJ}(\theta, \varphi) + \xi_{n'n} \frac{1}{r} g_{n'l+\varepsilon t} \Phi_{l+\varepsilon MJ}.$$

Les $\xi_{n'n}$ sont largement plus petits que 1, comme nous l'avons vu dans l'exemple calculé à la Section 7. Pour simplifier, nous les prenons tous égaux, au plus grand par exemple; cela ne change pas le signe de la différence $(N^+ - N^-)$ qui seul importe. En posant $\nu = (nlMJt)$, $\nu' = (n, l+\varepsilon, M, J, t)$, et indiquant la parité de $\zeta_\nu(\mathbf{x}) = (1/r) g_{nlt} \Phi_{lMJ}$ par un indice $+$ ou $-$, on a:

Pour l pair $\qquad \chi_{nlMJt}(\mathbf{x}) = \chi_\nu \simeq \zeta_\nu^+ + \xi \zeta_{\nu'}^-$,

l impair $\qquad \chi_{nlMJt}(\mathbf{x}) = \chi_\nu \simeq \zeta_\nu^- + \xi \zeta_{\nu'}^+$.

La norme des ζ_ν est égale à 1.

Remplissons un niveau de $(nlJt)$ déterminés, et appelons $\varphi^\pm(k)$ les fonctions correspondant a k particules sur le niveau:

a) l pair:

1 particule: $\qquad \varphi^+(1) \simeq \zeta_{\nu_1}^+, \qquad \varphi^-(1) \simeq \xi \zeta_{\nu_1}^-,$

$$N^+(1) \simeq 1, \qquad N^-(1) \simeq \xi^2;$$

récurence:
$$\varphi^+(k+1) \simeq \zeta_{\nu_{k+1}}^+ \varphi^+(k) + \xi \zeta_{\nu'_{k+1}}^- \varphi^-(k),$$
$$\varphi^-(k+1) \simeq \xi \zeta_{\nu'_{k+1}}^- \varphi^+(k) + \zeta_{\nu_{k+1}}^+ \varphi^-(k).$$

D'où:

$$N^+(k+1) \simeq N^+(k) \simeq 1, \qquad N^-(k+1) \simeq N^-(k) \simeq \xi^2.$$

Donc, pour une couche de l pair: $N^+ \simeq 1$, $N^- \simeq \xi^2$, quel que soit $k \leqslant 2J+1$.

b) l impair:

1 particule: $\varphi^+(1) \simeq \xi \zeta^+_{\nu'_1}$, $\varphi^-(1) \simeq \zeta^-_{\nu_1}$,

$N^+(1) \simeq \xi^2$, $N^-(1) \simeq 1$;

récurence: $\varphi^+(k+1) \simeq \xi \zeta^+_{\nu'_{k+1}} \varphi^+(k) + \zeta^-_{\nu_{k+1}} \varphi^-(k)$,

$\varphi^-(k+1) \simeq \zeta^-_{\nu_{k+1}} \varphi^+(k) + \xi \zeta^+_{\nu'_{k+1}} \varphi^-(k)$.

D'où:

$$N^+(k+1) \simeq N^-(k), \qquad N^-(k+1) \simeq N^+(k),$$

et:

$N^+(k) \simeq 1$, $N^-(k) \simeq \xi^2$ si k est pair,

$N^-(k) \simeq 1$, $N^+(k) \simeq \xi^2$ si k est impair.

Pour un niveau plein, $N^+ \simeq 1$ et $N^- \simeq \xi^2$ car $2J+1$ est pair.

La généralisation à un cœur formé de couches complètes plus une couche incomplète est immédiate; le cœur ne joue pas de rôle puisque dans son cas $(N^+ - N^-) \simeq 1 - \xi^2 > 0$; seule importe la parité du l de la couche incomplète, et de son occupation k. Si l'on suppose maintenant que $h > 0$ et utilise les résultats ci-dessus, on en déduit le tableau comparatif suivant, des parités calculées dans les modèles en couches ordinaire et pseudoscalaire:

	Modèle en couches ordinaire	Modèle pseudo-scalaire
Toutes les couches complètes	Parité $= +1$	$\Delta E \simeq h(1-\zeta^2) > 0$, état fondamental de parité $= +1$
Cœur $+ k$ particules sur niveau à l pair	Parité $= +1$	$\Delta E > 0$, parité $= +1$
Cœur $+ k$ particules sur niveau à l impair	Parité $= (-1)^k$	$\Delta E \simeq h(1-\zeta^2)(-1)^k$, parité $= (-1)^k$

La parité obtenue dans le modèle pseudo-scalaire est bien la même que dans le cas du modèle en couches habituel, sous réserve que $h > 0$, ce que nous allons démontrer.

Lorsqu'on ne suppose rien sur les normes individuelles de $\varphi^\pm$ et $|\pm\rangle$, on a:

$$h = \frac{\langle - | \boldsymbol{H}^0_M | - \rangle}{\langle - | - \rangle} - \frac{\langle + | \boldsymbol{H}^0_M | + \rangle}{\langle + | + \rangle}.$$

Jusqu'ici nous avons laissé les kets $|\pm\rangle$ indéterminés, et travaillé directement avec $\langle M|\boldsymbol{V}_\alpha|M\rangle$ qui est donné par une équation aux sources (2.4), les fonctions $\varphi^\pm$ étant inconnues. Le problème est maintenant inversé, et les $|\pm\rangle$ sont les inconnues à calculer.

En introduisant $\Psi = \varphi(X,\eta)\cdot|M\rangle$ dans (1.2), multipliant chaque membre par $\varphi^*(X,\eta)$ et intégrant, on obtient:

$$(\boldsymbol{H}^0_M + \boldsymbol{H}'_M)|M\rangle = E_M|M_M\rangle\,. \tag{9.2}$$

Nous prenons pour les vecteurs $|+\rangle$ et $|-\rangle$ la même approximation que pour φ^+ et φ^-, et posons: $|+\rangle =$ partie paire de $|M\rangle$, $|-\rangle =$ partie impaire de $|M\rangle$. Nous sommes en présence d'un problème résoluble exactement: la description des mésons virtuels engendrés par A nucléons dont le mouvement est connu et donné par les fonctions $\varphi^\pm$, calculées comme à la Section 7.

Le terme d'interaction peut être transformé en intégrant par parties:

$$\boldsymbol{H}'_M = (\varphi(x,\eta), \boldsymbol{H}'\varphi(x,\eta)) = \frac{f}{\mu}\int \mathrm{d}^3x \sum_{a=1}^{A} \chi_a^*(\mathbf{x})\boldsymbol{\sigma}_a\tau_a^\alpha\chi_a(\mathbf{x})\boldsymbol{\nabla}\cdot\boldsymbol{V}_\alpha(\mathbf{x}) =$$

$$= -\frac{f}{\mu}\sum_{a=1}^{A}\int \mathrm{d}^3x\,\boldsymbol{V}_\alpha(\mathbf{x})\boldsymbol{\nabla}(\chi_a^*\boldsymbol{\sigma}\tau\chi_a)\,.$$

Posons:

$$\begin{cases} \varrho_a^\alpha = \boldsymbol{\nabla}(\chi_a^*\boldsymbol{\sigma}\tau^\alpha\chi_a)\,, & \boldsymbol{V} = \dfrac{1}{\sqrt{2}}(\boldsymbol{V}_1 + i\boldsymbol{V}_2)\ (^*)\,, \\ \varrho^3(\mathbf{x}) = \displaystyle\sum_{a=1}^{A}\varrho_a^3\,, & \varrho(\mathbf{x}) = \displaystyle\sum_{a=1}^{A}\frac{1}{\sqrt{2}}(\varrho_a^1 + i\varrho_a^2)\,. \end{cases} \tag{9.3}$$

Il vient:

$$\boldsymbol{H}'_M = -\frac{f}{\mu}\int \mathrm{d}^3x(\varrho^3\boldsymbol{V}_3 + \varrho\boldsymbol{V} + \varrho^*\boldsymbol{V}^*)\,. \tag{9.4}$$

En introduisant dans (9.4) les formes explicites de $\boldsymbol{V}_3$ et $\boldsymbol{V}$ (*), puis les transformées de ϱ^3 et ϱ relativement à la base $\mathscr{Y}_{lm}$:

$$u_{lm}(k) = \int \mathrm{d}^3x\,\varrho^3\,\mathscr{Y}_{lm}(k,\mathbf{x})\,,\quad v_{lm}(k) = \int \mathrm{d}^3x\,\varrho\,\mathscr{Y}_{lm}(k,\mathbf{x})\,,$$

$$w_{lm}(k) = \int \mathrm{d}^3x\,\varrho^*\,\mathscr{Y}_{lm}(k,\mathbf{x})\,, \tag{9.5}$$

(*) Voir Appendice.

on trouve:

$$\boldsymbol{H}'_M = -\frac{f}{\mu} \sum_{\substack{l=0 \\ -l\leq m\leq l}}^{\infty} \int_0^{\infty} \frac{\mathrm{d}k}{\sqrt{2\omega_k}} \{\boldsymbol{a}_3(k,l,m)\, u_{lm}(k) + \boldsymbol{a}_3^* u_{lm}^* + \boldsymbol{a}_2 v_{lm} + \boldsymbol{a}_2^* v_{lm}^* + \\ + \boldsymbol{a}_1 w_{lm} + \boldsymbol{a}_1^* w_{lm}^*\} . \tag{9.6}$$

D'autre part:

$$\boldsymbol{H}'_M = \sum_{\substack{l,m \\ \alpha=1,2,3}} \int_0^{\infty} \mathrm{d}k\, \omega_k \boldsymbol{a}_\alpha^*(k,l,m) \boldsymbol{a}_\alpha(k,l,m) .$$

L'hamiltonien $\boldsymbol{H}_M = \boldsymbol{H}^0_M + \boldsymbol{H}'_M$ peut être diagonalisé au moyen de la transformation unitaire $U = \exp[iS]$, où

$$S = \frac{i}{\sqrt{2}} \frac{f}{\mu} \sum_{l,m} \int_0^{\infty} \mathrm{d}k\, \omega_k^{-\frac{3}{2}} (\boldsymbol{a}_3 u - \boldsymbol{a}_3^* u^* + \boldsymbol{a}_2 v - \boldsymbol{a}_2^* v^* + \boldsymbol{a}_1 w - \boldsymbol{a}_1^* w^*) .$$

On vérifie aisément que:

$$[S, \boldsymbol{H}^0_M] = -i\boldsymbol{H}'_M , \qquad [S, \boldsymbol{H}'_M] = -i2\varepsilon , \qquad [S, \varepsilon] = 0 ,$$

où

$$\varepsilon = \frac{1}{2}\left(\frac{f}{\mu}\right)^2 \sum_{l,m} \int_0^{\infty} \frac{\mathrm{d}k}{\omega_k^2} (u^* u + v^* v + w^* w) .$$

En conséquence:

$$U^{-1} \boldsymbol{H}_M U = \boldsymbol{H}^0_M - \varepsilon . \tag{9.7}$$

Et le vecteur d'état solution de (9.2) est:

$$|M\rangle = \exp[iS]|0\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!} S^n |0\rangle . \tag{9.8}$$

En supposant, comme nous avons fait jusqu'ici, que $\varrho^\alpha = \varrho^\alpha(r)$, les seules transformées (9.5) non nulles sont celles d'indices $l = 0$, et $m = 0$. $|M\rangle$ ne contient plus que les mésons S, et ses parties paire et impaire sont:

$$|+\rangle = \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} S^{2n}|0\rangle , \qquad |-\rangle = \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} S^{2n+1}|0\rangle , \tag{9.9}$$

$S^{2n}|0\rangle$ est une somme de vecteurs contenant un nombre pair de mésons, et $S^{2n+1}|0\rangle$ un nombre impair.

En vertu de l'unitarité de U et de l'orthogonalité de $|+\rangle$ et $|-\rangle$, on a:

$$\langle +|+\rangle = \langle M|M\rangle - \langle -|-\rangle = 1 - \langle -|-\rangle .$$

D'autre part:

$$\langle +|\boldsymbol{H}^0_M|+\rangle = \varepsilon - \langle -|\boldsymbol{H}^0_M|-\rangle ,$$

car

$$\langle M|\boldsymbol{H}^0_M|M\rangle = \langle 0|U^{-1}\boldsymbol{H}^0_M U|0\rangle = \varepsilon , \quad \text{et} \quad \langle -|\boldsymbol{H}^0_M|+\rangle = 0 .$$

D'où:

$$h = \frac{\langle -|\boldsymbol{H}^0_M|-\rangle - \varepsilon\langle -|-\rangle}{\langle -|-\rangle(1-\langle -|-\rangle)} , \tag{9.10}$$

avec

$$\langle -|-\rangle = \sum_{n=0}^{\infty}(-1)^N\langle 0|S^{2(N+1)}|0\rangle \cdot \sum_{n+n'=N}\frac{1}{(2n+1)!\,(2n'+1)!} ;$$

$$\langle -|\boldsymbol{H}^0_M|-\rangle = \frac{1}{2}\sum_{N=0}^{\infty}(-1)^N \sum_{n+n'=N}\frac{1}{(2n+1)!\,(2n'+1)!}\cdot$$
$$\cdot\langle 0|S^{2n+1}\boldsymbol{H}^0_M S^{2n'+1} + S^{2n'+1}\boldsymbol{H}^0_M S^{2n+1}|0\rangle .$$

Le calcul de $\langle -|\boldsymbol{H}^0_M|-\rangle$ se ramène à celui des éléments de matrice $\langle 0|S^{2N}|0\rangle$. En vertu de $[S, \boldsymbol{H}^0_M] = -i\boldsymbol{H}'_M$ et $[S^m, \boldsymbol{H}'_M] = -i2m\varepsilon S^{m-1}$ on a:

$$S^{2n+1}\boldsymbol{H}^0_M S^{2n'+1} + S^{2n'+1}\boldsymbol{H}^0_M S^{2n+1} = S^{2n+2}\boldsymbol{H}^0_M S^{2n'} +$$
$$+ S^{2n'}\boldsymbol{H}^0_M S^{2n+2} + 2(2n-2n'+1)\varepsilon S^{2(n+n')} .$$

On démontre alors aisément que:

$$S^{2n+1}\boldsymbol{H}^0_M S^{2n'+1} + S^{2n'+1}\boldsymbol{H}^0_M S^{2n+1} = S^{2(n+n'+1)}\boldsymbol{H}^0_M +$$
$$+ \boldsymbol{H}^0_M S^{2(n+n'+1)} + 2(2n+1)(2n'+1)\varepsilon S^{2(n+n')} .$$

Comme $\boldsymbol{H}^0_M|0\rangle = 0$, on a: $\langle 0|\boldsymbol{H}^0_M S^{2(n+n'+1)}|0\rangle = \langle 0|S^{2(n+n'+1)}\boldsymbol{H}^0_M|0\rangle = 0$; et finalement:

$$\langle -|\boldsymbol{H}^0_M|-\rangle = \varepsilon\sum_{N=0}^{\infty}(-1)^N\langle 0|S^{2N}|0\rangle \cdot \sum_{n+n'=N}\frac{1}{(2n)!\,(2n')!}\cdot$$

Pour simplifier, calculons $\langle 0|S^{2N}|0\rangle$ dans le cas où les mésons chargés sont absents; il reste $S = (i/\sqrt{2})(f/\mu)\int \mathrm{d}k\,\omega_k^{-\frac{3}{2}}(\boldsymbol{a}(k) - \boldsymbol{a}^*(k))\,u(k)$ (en supprimant les

indices), $\varrho^3 = \varrho^3(r)$ entraînant $u_{lm}(k) = \delta_{l0}\delta_{m0}u(k)$. La généralisation au cas général est quasi évidente.

$$\langle 0|S^{2N}|0\rangle = \left(\frac{i}{\sqrt{2}}\right)^{2N}\left(\frac{f}{\mu}\right)^{2N}\int\cdots\int\frac{\mathrm{d}k_1\ldots\mathrm{d}k_{2N}}{(\omega_{k_1}\ldots\omega_{k_{2N}})^{\frac{3}{2}}}\langle 0|\prod_{i=1}^{2N}(\boldsymbol{a}(k_i)-\boldsymbol{a}^*(k_i))|0\rangle u(k_1)\ldots u(k_{2N}).$$

Posons:

$$\boldsymbol{a}_i^{\nu} = \begin{cases} \boldsymbol{a}_i = \boldsymbol{a}(k_i)\,, & \nu = 0\,, \\ \boldsymbol{a}_i^* = \boldsymbol{a}^*(k_i)\,, & \nu = 1\,, \end{cases} \qquad i = 1, \ldots, 2N.$$

On peut alors écrire:

$$\langle 0|\prod_{i=1}^{2N}(\boldsymbol{a}(k_i) - \boldsymbol{a}^*(k_i))|0\rangle = \sum_{\nu_1,\ldots,\nu_{2N}=0}^{1}(-1)^{\nu_1+\ldots+\nu_{2N}}\langle 0|\boldsymbol{a}_1^{\nu_1}\ldots\boldsymbol{a}_{2N}^{\nu_{2N}}|0\rangle\,.$$

Les relations de commutation se résument à:

$$[\boldsymbol{a}_i^{\nu}, \boldsymbol{a}_{i'}^{\nu'}] = (\nu - \nu')\,\delta(k_i - k_{i'})\,.$$

Tous les termes se terminant à droite par $a_{2N}^{\nu=0} = \boldsymbol{a}_{2N}$ donnent zéro puisque $\boldsymbol{a}(k)|0\rangle = 0$. De plus, tous les termes tels que $\nu_1+\ldots+\nu_{2N} \neq N$ donnent également zéro car ils contiennent des nombres inégaux d'opérateurs de création et d'annihilation. On calcule la contribution de chaque terme restant en appliquant les règles de commutation; on prend le premier opérateur $\boldsymbol{a}_i$ à partir de la droite, et on le déplace jusqu'à ce qu'il soit devant le ket $|0\rangle$; à chaque commutation avec un $\boldsymbol{a}_l^*$ on obtient un nouveau terme contenant une fonction $\delta(k_i - k_l)$ et deux opérateurs de moins. On procède de même jusqu'à ce qu'on n'ait plus que des termes donnant zéro et des produits de N fonctions δ. Jamais deux fonctions des deux mêmes arguments n'apparaissent dans un même produit car chaque $\boldsymbol{a}^{\nu}$ n'apparaît qu'une fois dans un terme. De plus aucun des termes de départ ne peut s'obtenir d'un autre par permutation des opérateurs. Tous les produits de δ seront différents. En outre l'intégrale multiple est symétrique par rapport à toutes les variables k_i. Le résultat est donc forcément:

$$(9.11) \qquad (-1)^N \sum_{\text{perm } k_1\ldots k_{2N}} \delta(k_1 - k_2)\,\delta(k_3 - k_4)\ldots\delta(k_{2N-1} - k_{2N})\,.$$

Si p_N est le nombre de termes de la somme (9.11) pour N, et p_{N-1} pour $N-1$, on a la relation: $p_N = (2N-1)p_{N-1}$; d'où:

$$p_N = (2N-1)!! \qquad (N \geqslant 2).$$

Finalement, chaque terme donnant la même contribution, on a:

$$\langle 0|S^{2N}|0\rangle = (2N-1)!!\left[\frac{1}{2}\left(\frac{f}{\mu}\right)^2\int_0^\infty \frac{dk}{\omega_k^3}\,u^2(k)\right]^N. \tag{9.12}$$

Si on prend les mésons chargés:

$$\langle 0|S^{2N}|0\rangle = (2N-1)!!\left[\frac{1}{2}\left(\frac{f}{\mu}\right)^2\int\frac{dk}{\omega_k^3}\cdot\right.$$
$$\left.\cdot\{u^2(k)+v(k)v^*(k)+w(k)w^*(k)\}\right]^N = (2N-1)!!\cdot I^N \tag{9.12'}$$

avec $v(k)=v_{00}(k)$, $w(k)=w_{00}(k)$.

Substituons ces valeurs dans les développements donnant $\langle -|\boldsymbol{H}_M^0|-\rangle$ et $\langle -|-\rangle$:

$$\langle -|\boldsymbol{H}_M^0|-\rangle = \varepsilon\left(1+\sum_{N=0}^{\infty}(-1)^N(2N-1)!!\cdot I^{N+1}\cdot\sum_{n+n'=N}\frac{1}{(2n)!\,(2n')!}\right),$$

$$\langle -|-\rangle = \sum_{N=0}^{\infty}(-1)^N(2N+1)!!\cdot I^{N+1}\cdot\sum_{n+n'=N}\frac{1}{(2n+1)!\,(2n'+1)!}.$$

En vertu de

$$\sum_{n+n'=N}\frac{1}{(2n)!\,(2n')!} = \sum_{n+n'=N-1}\frac{1}{(2n+1)!\,(2n'+1)!} = \frac{1}{2}\,\frac{2^{2N}}{(2N)!},$$

on a

$$\begin{cases}(-1)^N(2N-1)!!\cdot I^N\cdot\displaystyle\sum_{n+n'=N}\frac{1}{(2n)!\,(2n')!} = \frac{1}{2}\,\frac{(-1)^N}{N!}\,(2I)^N,\\[2ex] (-1)^N(2N+1)!!\cdot I^{N+1}\cdot\displaystyle\sum_{n+n'=N}\frac{1}{(2n+1)!\,(2n'+1)!} = -\frac{1}{2}\,\frac{(-1)^{N+1}}{(N+1)!}\,(2I)^{N+1}.\end{cases} \tag{9.13}$$

C'est, au $\frac{1}{2}$ près, le $N^{\text{ième}}$ terme du développement en série de $\exp[-2I]$. On a ainsi, simplement:

$$\langle -|\boldsymbol{H}_M^0|-\rangle = \tfrac{1}{2}\varepsilon(1+\exp[-2I]),\quad \langle +|\boldsymbol{H}_M^0|+\rangle = \tfrac{1}{2}\varepsilon(1-\exp[-2I]), \tag{9.14}$$

$$\langle -|-\rangle = \tfrac{1}{2}(1-\exp[-2I]),\qquad \langle +|+\rangle = \tfrac{1}{2}(1+\exp[-2I]). \tag{9.15}$$

Et:

$$h = \varepsilon\,\frac{4\exp[-2I]}{1-\exp[-4I]}. \tag{9.16}$$

Conclusion: $h>0$, puisque $\varepsilon>0$ et $I>0$.

Evaluation de h. – Calculons d'abord ε, dans le cas $\varrho^\alpha = \varrho^\alpha(r)$.

D'après la définition (9.5), $v(k) = w^*(k) = \int \mathrm{d}^3x \left(\varrho^1(r) + i\varrho^2(r)\right) \mathscr{Y}_{00}(k, \mathbf{x})$, car $\mathscr{Y}^*_{00} = \mathscr{Y}_{00} = kj_0(kr) Y_{00}(\theta, \varphi) = (4\pi)^{-\frac{1}{2}} kj_0(kr)$. En posant: $u_\alpha(k) = \int \mathrm{d}^3x \, \varrho^\alpha(r) \mathscr{Y}_{00}(k, \mathbf{x})$ on a:

$$\varepsilon = \frac{1}{2}\left(\frac{f}{\mu}\right)^2 \sum_{\alpha=1}^{3} \int_0^\infty \frac{\mathrm{d}k}{k^2 + \mu^2} \, u_\alpha^2(k) \,. \tag{9.17}$$

L'intégrale se calcule facilement, et on trouve finalement:

$$\varepsilon = 2\pi \frac{f}{\mu} \sum_{\alpha=1}^{3} \int_0^\infty \mathrm{d}r r^2 \varrho^\alpha(r) V_\alpha(r) \,. \tag{9.18}$$

A titre d'exemple, calculons ε avec les valeurs trouvées à la Section **7**; introduisons dans (9.18) le $\varrho(r)$ donné en (7.7), en prenant garde au fait qu'il contient déjà f/μ. L'intégration est immédiate en vertu de (7.9). On trouve:

$$\varepsilon = U_0 \frac{12}{p^2} \left(10 Y_1 \xi_1 - 3 Y_2 \xi_2 + 3 Y_3 \xi_3 + 4 Y_4 \xi_4\right) \simeq 118 \text{ MeV} \,.$$

Puisque nous calculons h uniquement pour avoir une idée de son ordre de grandeur, nous pouvons nous permettre d'évaluer I au moyen d'une approximation grossière. La transformée $u_\alpha(k)$ est vraisemblablement déjà très petite quand $k = \mu$ $(\omega_k = 2\mu)$; aussi posons-nous:

$$(k^2 + \mu^2)^{-\frac{3}{2}} \simeq \mu^{-1}(k^2 + \mu^2)^{-1} \,.$$

De sorte que:

$$I \simeq \frac{1}{\mu} \varepsilon \,.$$

Et

$$h \simeq \varepsilon \frac{4 \exp\left[-2\varepsilon/\mu\right]}{1 - \exp\left[-4\varepsilon/\mu\right]} \simeq 80 \text{ MeV} \,.$$

Le ΔE est plus petit que h à cause du facteur $|N^+ - N^-| < 1$. Pour ne ne pas avancer de nombre sans justification, disons que sa valeur se mesurera en dizaines de MeV.

Quelques valeurs limite.

1) $\lim_{f\to 0} I = \lim_{f\to 0} \varepsilon = 0$, $\lim_{f\to 0} h = \mu$, $\lim_{f\to 0} \langle + | + \rangle = 1$, $\lim_{f\to 0} \langle - | - \rangle = 0$.

Si le couplage s'évanouit (constante de couplage $f \to 0$), le vecteur $|M\rangle$ tend vers $|0\rangle$ qui est de parité (+). Mais les niveaux (+) et (—) ne se réunissent pas car $h \to \mu$, de même que ΔE; la fonction $\varphi(X, \eta)$ tend vers une fonction à parité déterminée.

2) $$\lim_{f\to\infty} I = \lim_{f\to\infty} \varepsilon = \infty\,, \quad \lim_{f\to\infty} h = 0\,, \quad \lim_{f\to\infty} \langle + | + \rangle = \lim_{f\to\infty} \langle - | - \rangle = \tfrac{1}{2}\,.$$

Si le couplage croît fortement, le nombre des mésons a tendance à s'équilibrer entre les vecteurs $|+\rangle$ et $|-\rangle$, et les états (+) et (—) tendent à se rapprocher:

$$\Delta E(f = 0) = 135 \text{ MeV}\,, \quad \Delta E(f = 1.1) < 80 \text{ MeV}\,, \quad \Delta E(f = \infty) = 0\,.$$

Démonstration de $(\varphi^-, H'_I\varphi^+) - (\varphi^+, H'_I\varphi^-) = 0$.

$$V^I_\alpha = \frac{1}{2i}\left(\langle + | \boldsymbol{V}_\alpha | - \rangle - \langle - | \boldsymbol{V}_\alpha | + \rangle\right),$$

(9.19) $$\langle + | \boldsymbol{V}_\alpha | - \rangle = i \sum_{n,n'=0}^{\infty} \frac{(-1)^{n+n'}}{(2n)!\,(2n'+1)!} \langle 0 | S^{2n} \boldsymbol{V}_\alpha S^{2n'+1} | 0 \rangle\,,$$

(9.20) $$[S, \boldsymbol{a}_\alpha(k, l, m)] = [S, \boldsymbol{a}^*_\alpha(k, l, m)] = \delta_{l0}\delta_{m0} \frac{i}{\sqrt{2}} \frac{f}{\mu} \frac{u_\alpha(k)}{\omega_k^{\frac{1}{2}}}\,,$$

où S est défini en (9.12′).

Soit

$$\boldsymbol{W}_\alpha = \sum_{l,m} \int \frac{\mathrm{d}k}{2\omega_k} \boldsymbol{a}_\alpha(k, l, m)\, \mathscr{Y}_{lm}(k, \mathbf{x})\,;$$

le commutateur de S et $\boldsymbol{W}_\alpha$ vaut, en vertu de (9.20):

$$[S, \boldsymbol{W}_\alpha] = i\,\frac{1}{2}\left(\frac{f}{\mu}\right)^2 \int_0^\infty \frac{k^2\,\mathrm{d}k}{\omega_k^2}\, u_\alpha(k) j_0(kr) = iK_\alpha(r)\,.$$

D'où:

$$[S^m, \boldsymbol{W}_\alpha] = im K_\alpha(r) S^{-1}\,.$$

Ce qui permet de dire:

$$i\langle 0 | S^{2n} \boldsymbol{W}_\alpha S^{2n'+1} | 0 \rangle = (2n'+1) K_\alpha(r) \langle 0 | S^{2(n+n')} | 0 \rangle = \text{fonction réelle};$$

de même pour $\boldsymbol{W}^*_\alpha$. En conclusion, $\langle + | \boldsymbol{V}_\alpha | - \rangle$ est réel, puisque $\boldsymbol{V}_\alpha$ est une combinaison linéaire des $W_{\alpha'}$ et $W^*_{\alpha'}$, et $V^I_\alpha(r) = H'_I = 0$.

10. – Utilisation des fonctions d'onde; moments magnétiques.

Il existe trois types d'opérateurs dont on peut avoir besoin de calculer la valeur moyenne pour un état Ψ du noyau.

1) Opérateurs $\boldsymbol{O}_{\mathcal{N}}$ n'agissant que sur la partie nucléonique de la fonction d'onde $\Psi^p = \varphi^+|p\rangle + \varphi^-|q\rangle$ $(p = \pm,\ q = \mp)$.

$$\langle\Psi^p|\boldsymbol{O}_{\mathcal{N}}|\Psi^p\rangle = \langle p|p\rangle(\varphi^+, \boldsymbol{O}_{\mathcal{N}}\varphi^+) + \langle q|q\rangle(\varphi^-, \boldsymbol{O}_{\mathcal{N}}\varphi^-)\,,$$

car $\boldsymbol{O}_{\mathcal{N}}\varphi(x,\eta)\cdot|M\rangle = |M\rangle\boldsymbol{O}_{\mathcal{N}}\varphi$; de plus, on peut choisir $\langle p|p\rangle = \langle q|q\rangle = 1$, d'où:

$$\langle\Psi^p|\boldsymbol{O}_{\mathcal{N}}|\Psi^p\rangle = \sum_{\eta}\int_{X}\varphi^{*+}(X,\eta)\boldsymbol{O}_{\mathcal{N}}\varphi^+(X,\eta) + \varphi^{*-}(X,\eta)\boldsymbol{O}_{\mathcal{N}}\varphi^-(X,\eta). \tag{10.1}$$

En particulier, du point de vue du modèle en couches, on peut s'intéresser à un opérateur $\boldsymbol{O}^a_{\mathcal{N}}$ n'agissant que sur la fonction χ_a du nucléon d'état $(n_a l_a M_a J_a t_a)$; en laissant de côté l'antisymétrisation, on a:

$$\varphi^{\pm}_{(A)} = \varphi^{+}_{(A-1)}\chi^{\pm}_a + \varphi^{-}_{(A-1)}\chi^{\mp}_a\,,$$

où $\varphi^{\pm}_{(A-1)}$ sont les fonctions combinées relatives au $A-1$ nucléons restants, et $\chi^{\pm}_a$ les parties paire et impaire de χ_a. Avec une normalisation adéquate des fonctions, (10.1) donne dans ce cas:

$$\langle\Psi|\boldsymbol{O}^a_{\mathcal{N}}|\Psi\rangle = \int \mathrm{d}^3x(\chi^{*+}_a\boldsymbol{O}^a_{\mathcal{N}}\chi^+_a + \chi^{*-}_a\boldsymbol{O}^a_{\mathcal{N}}\chi^-_a)\,. \tag{10.2}$$

On remarque qu'il n'est pas nécessaire de connaître les kets $|p\rangle$ et $|q\rangle$.

2) Opérateurs $\boldsymbol{O}_M$ n'agissant que sur la partie mésonique de Ψ.

$$\langle\Psi|\boldsymbol{O}_M|\Psi\rangle = (\varphi^+, \varphi^+)\langle p|\boldsymbol{O}_M|p\rangle + (\varphi^-, \varphi^-)\langle q|\boldsymbol{O}_M|q\rangle\,.$$

3) Opérateurs $\boldsymbol{O}$ agissant sur les deux parties.

$$\langle\Psi|\boldsymbol{O}|\Psi\rangle = (\varphi^+, \Omega^p\varphi^+) + (\varphi^-, \Omega^q\varphi^-) + (\varphi^+, \Omega\varphi^-) + (\varphi^-, \Omega^*\varphi^+)\,,$$

où $\Omega^p = \langle p|\boldsymbol{O}|p$, $\Omega^q = \langle q|\boldsymbol{O}|q\rangle$, $\Omega = \langle p|\boldsymbol{O}|q\rangle$.

Les opérateurs en relation avec la parité et la charge, en particulier, sont de ce type.

Moments magnétiques. – La formule (10.2) permet une constatation intéressante, relative au moment magnétique d'un noyau calculé dans le modèle pseudo-scalaire. Si l'on suppose que le moment magnétique est entièrement dû aux nucléons, on tire facilement de (10.2) que la contribution des couches pleines est nulle. Ainsi, le modèle prévoit que les noyaux magiques ont un moment magnétique nul. Pour un noyau constitué d'un cœur fermé plus un nucléon:

$$\langle\mu\rangle = \int \mathrm{d}^3x(\chi_A^{*+}\mu\chi_A^+ + \chi_A^{*-}\mu\chi_A^-) = (\chi_A^+, \mu\chi_A^+) + (\chi_A^-, \mu\chi_A^-) .$$

χ_+^A et χ_A^- sont fonctions propres de $\mathbf{J}^2$, et de $\mathbf{L}^2$ pour les valeurs propres différant de 1: $l = J_A \pm \frac{1}{2}$. Ainsi, avec le modèle en couches ordinaire on aurait pour $\langle\mu\rangle$ soit $\mu^+ = |\chi_A^+|^{-2}(\chi_A^+, \mu\chi_A^+)$, soit $\mu^- = |\chi_A^-|^{-2}(\chi_A^-, \mu\chi_A^-)$, suivant que $l = J_A \pm \frac{1}{2}$ est pair ou impair. Avec le modèle pseudo-scalaire on a:

$$\langle\mu\rangle = |\chi_A^+|^2 \cdot \mu^+ + |\chi_A^+|^2 \cdot \mu^- \qquad (|\chi_A^-|^2 + |\chi_A^-|^2 = 1).$$

Le moment magnétique se trouve entre les lignes de Schmidt.

* * *

En terminant, je tiens à remercier le Professeur K. Bleuler qui m'a suggéré le sujet de ce travail. Ses directives éclairées et ses conseils me furent constamment d'une aide précieuse.

Appendice

A) *Unités et symboles.*

Les unités utilisées sont telles que $\hbar = c = 1$.

f = constante de couplage;
μ = masse d'un méson π;
m = masse d'un nucléon;
$A = Z + N$ = nombre de nucléons;
$X = (\mathbf{x}_1, \ldots, \mathbf{x}_A)$ = ensemble des coordonnées spatiales des A nucléons;
$\eta = (s_1 t_1, \ldots, s_A t_A)$ = ensemble des coordonnées de spin et iso-spin;
α = indice d'iso-spin. Toute sommation sur α est sous-entendue.
* désigne le conjugué hermitien;
$\pm$ désignent la parité lorsqu'ils sont utilisés comme indice;
$\sim$ est réservé à distinguer deux grandeurs différentes, désignées par la même lettre.

Les opérateurs agissant sur l'espace de Hilbert nucléons-mésons π sont désignés par des lettres italiques grasses: $\boldsymbol{H}$, $\boldsymbol{V}_\alpha(\mathbf{x})$, $\boldsymbol{\Pi}$, ... Les grandeurs se transformant comme des vecteurs de l'espace euclidien à trois dimensions sont notés en lettres grasses: $\mathbf{x}$, $\boldsymbol{\nabla}$, $\boldsymbol{\sigma}$, ... Les éléments de matrice et produits hermitiens sont parfois notés de deux différentes manières suivant la forme explicite de l'espace de Hilbert:

$$\begin{matrix} \langle M_1 | \boldsymbol{H} | M_2 \rangle, \\ \langle M_1 | M_2 \rangle, \end{matrix} \quad \text{ou} \quad \begin{matrix} (\varphi_1, \boldsymbol{H}\varphi_2) \\ (\varphi_1, \varphi_2) \, . \end{matrix}$$

B) *Champs d'opérateurs mésoniques* $\boldsymbol{V}_\alpha(\mathbf{x})$.

Les opérateurs $\boldsymbol{V}_\alpha$ sont exprimés en représentation de Schrödinger, dans un système de coordonnées sphériques $(\mathbf{x} = (r, \theta, \varphi))$:

$$\boldsymbol{V}_3(\mathbf{x}) = \sum_{\substack{l=0 \\ -l \leq m \leq l}} \int_0^\infty \frac{\mathrm{d}k}{\sqrt{2\omega_k}} \left(\boldsymbol{a}_3(k, l, m) \cdot \mathscr{Y}_{lm}(k, \mathbf{x}) + \boldsymbol{a}_3^*(k, l, m) \cdot \mathscr{Y}_{lm}^*(k, \mathbf{x}) \right) ,$$

$$\boldsymbol{V}(\mathbf{x}) = \frac{1}{\sqrt{2}} (\boldsymbol{V}_1 + i\boldsymbol{V}_2) = \sum_{l,m} \int_0^\infty \frac{\mathrm{d}k}{\sqrt{2\omega_k}} \left(\boldsymbol{a}_1(k, l, m) \cdot \mathscr{Y}_{lm}(k, \mathbf{x}) + \boldsymbol{a}_2^*(l, k, m) \cdot \mathscr{Y}_{lm}^*(k, \mathbf{x}) \right) .$$

Les fonctions de base $\mathscr{Y}_{lm}(k, \mathbf{x})$ sont, explicitement:

$$\mathscr{Y}_{lm}(k, \mathbf{x}) = k j_l(kr) \, Y_{lm}(\theta, \varphi) \, ,$$

où $j_l(kr) = \sqrt{(k/r)} J_{l+\frac{1}{2}}(kr)$, $J_{l+\frac{1}{2}}(z) =$ fonction de Bessel (JAHNKE et EMDE), $Y_{lm}(\theta, \varphi) =$ fonction sphérique normalisée, $\omega_k = \sqrt{k^2 + \mu^2} =$ energie d'un méson π, $l, m =$ nombres quantiques orbital et azimutal d'un méson π.

On a les relations d'orthogonalité:

$$\int \mathrm{d}^3x \, \mathscr{Y}_{lm}^*(k, \mathbf{x}) \, \mathscr{Y}_{l'm'}(k', \mathbf{x}) = \delta_{ll'} \delta_{mm'} \delta(k - k') \, ,$$

$$\sum_{\substack{l=0 \\ -l \leq m \leq l}}^\infty \int_0^\infty \mathrm{d}k \, \mathscr{Y}_{lm}^*(k, \mathbf{x}) \, \mathscr{Y}_{lm}(k, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \, .$$

Les opérateurs de création et d'annihilation $a_\alpha^*(k, l, m)$ et $a_\alpha(k, l, m)$ satisfont les relations de commutation:

$$[\boldsymbol{a}_\alpha(k, l, m), \boldsymbol{a}_{\alpha'}^*(k', l', m')] = \delta_{\alpha\alpha'} \delta_{ll'} \delta_{mm'} \delta(k - k') \, ,$$

$$[\boldsymbol{a}_\alpha(k, l, m), \boldsymbol{a}_{\alpha'}^*(k', l', m')] = 0 \, .$$

Les vecteurs mésoniques $|M\rangle$ sont des combinaisons linéaires de vecteurs

$|\ldots M^{\alpha}_{klm}\ldots\rangle$ à nombres d'occupation fixes:

$$\boldsymbol{a}^{*}_{\alpha}(k,l,m)\boldsymbol{a}_{\alpha}(k,l,m)\,|\ldots M^{\alpha}_{klm}\ldots\rangle = M^{\alpha}_{klm}\,|\ldots M^{\alpha}_{klm}\ldots\rangle\,.$$

C) *Parité* $\boldsymbol{\Pi}$.

L'opérateur de parité $\boldsymbol{\Pi}$ relatif au système nucléons-mésons a la propriété:

$$[\boldsymbol{\Pi}, \boldsymbol{H}] = [\boldsymbol{\Pi}, \boldsymbol{H}^{0}_{M}] = [\boldsymbol{\Pi}, \boldsymbol{H}^{0}_{N}] = [\boldsymbol{\Pi}, \boldsymbol{H}'] = 0\,.$$

$\boldsymbol{\Pi}$ se décompose en un produit d'opérateurs lorsqu'il agit sur un produit de fonctions des variables respectivement nucléoniques et mésoniques

$$\Psi = \varphi(X,\eta)\,|M\rangle\,.$$

On a:

$$\boldsymbol{\Pi} = \boldsymbol{\Pi}_{N}\boldsymbol{\Pi}_{M} = \boldsymbol{\Pi}_{M}\boldsymbol{\Pi}_{N}\,, \qquad \boldsymbol{\Pi}\Psi = \boldsymbol{\Pi}_{N}\varphi(X,\eta)\boldsymbol{\Pi}_{M}\,|M\rangle\,,$$

$$\boldsymbol{\Pi}_{N}\varphi(X,\eta) = \varphi(-X,\eta)\,, \qquad \boldsymbol{\Pi}_{M}\,|\ldots M^{\alpha}_{klm}\ldots\rangle = (-1)^{\sum_{\alpha' k' l' m'} (l'+1) M^{\alpha'}_{k'l'm'}}\,|\ldots M^{\alpha}_{klm}\ldots\rangle\,.$$

Propriété: $[\boldsymbol{\Pi}_{N}, \boldsymbol{H}] \neq 0$, $[\boldsymbol{\Pi}_{M}, \boldsymbol{H}'] \neq 0$, bien que $[\boldsymbol{\Pi}_{N}\boldsymbol{\Pi}_{M}, \boldsymbol{H}'] = 0$. Pour simplifier, $\boldsymbol{\Pi}_{N}$ est noté π lorsqu'il agit sur une fonction concernant un seul nucléon: $\pi\chi(\mathbf{x}) = \chi(-\mathbf{x})$.

D) *Quelques propriétés utilisées des éléments de matrice de* $\boldsymbol{V}_{\alpha}(\mathbf{x})$.

1) Soient $|m,+\rangle$ et $|n,-\rangle$ deux vecteurs quelconques mais de parité déterminée. Alors:

$$\langle +, m\,|\boldsymbol{V}_{\alpha}(\mathbf{x})\,|n,\ -\rangle = f_{\alpha}(\mathbf{x}) = \quad f_{\alpha}(-\mathbf{x}) \qquad \text{fonction paire de } \mathbf{x}\,.$$

$$\langle +, m\,|\boldsymbol{V}_{\alpha}(\mathbf{x})\,|m,\ +\rangle = g_{\alpha}(\mathbf{x}) = -\,g_{\alpha}(-\mathbf{x}) \qquad \text{fonction impaire}\,,$$

$$\langle -, n\,|\boldsymbol{V}_{\alpha}(\mathbf{x})\,|n,\ -\rangle = h_{\alpha}(\mathbf{x}) = -\,h_{\alpha}(-\mathbf{x}) \qquad \text{fonction impaire}\,.$$

2) Soient $|S_1\rangle$, $|S_2\rangle$, deux vecteurs relatifs à des mésons S exclusivement, c'est-à-dire que: $\boldsymbol{a}_{\alpha}(k,l,m)\,|S_i\rangle = 0$ si $l \neq 0$. Alors:

$$\langle S_1\,|\boldsymbol{V}_{\alpha}(\mathbf{x})\,|S_2\rangle = f_{\alpha}(|\mathbf{x}|)\,.$$

3) Soient $|+\rangle$, $|-\rangle$, deux vecteurs de parité déterminée et relatifs à des mésons S. Les propriétés 1) et 2) entraînent:

$$\langle +\,|\boldsymbol{V}_{\alpha}(\mathbf{x})\,|-\rangle = V_{\alpha}(r)\,, \qquad \langle \pm\,|\boldsymbol{V}_{\alpha}(\mathbf{x})\,|\pm\rangle = 0\,.$$

4) Si $\langle m\,|\boldsymbol{V}_{\alpha}(\mathbf{x})\,|n\rangle = V_{\alpha}(r)$, on a: $[\pi, \boldsymbol{\sigma}\cdot\boldsymbol{\nabla} V_{\alpha}(r)] \neq 0$.

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